Math 255: Spring 2016
Midterm 1

NAME: SOLUTIONS

Time: **50 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Problem	Value	Score
1	7	
2	7	
3	8	
4	8	
5	4	
6	8	
7	8	
TOTAL	50	

Problem 1 : (7 points)

a) (4 points) Compute gcd(252, 198). You may use any technique you like, but you must justify your answer.

Euclidean algorithm

$252 = 1 \cdot 198 + 54$

$198 = 3 \cdot 54 + 36$

$54 = 1 \cdot 36 + 18$

$36 = 2 \cdot 18$

$\gcd(252, 198) = 18$

b) (3 points) Based on your answer above, does the equation $252x + 198y = 9$ have solution(s) in the integers? Please justify with **one** sentence.

No, because 18 does not divide 9.

Problem 2 : (7 points)

a) (3 points) Give the definition of $a \equiv b \pmod{n}$.

$a \equiv b \bmod n$ if n divides $a-b$.

b) (4 points) Prove that if $a \equiv b \pmod{n}$, then also $b \equiv a \pmod{n}$.

$a \equiv b \bmod n \Rightarrow n \mid (a-b)$

$\Rightarrow \exists\, k \in \mathbb{Z}$ such that $a-b = nk$

$\Rightarrow b-a = n(-k)$

$\Rightarrow$ since $-k \in \mathbb{Z}$ as well, $n \mid (b-a)$

$\Rightarrow b \equiv a \bmod n$

Problem 3 : (8 points) Prove the following statement using induction:

$$\sum_{j=0}^{n} 2^j = 2^{n+1} - 1 \quad \text{for all } n \geq 1.$$

Base case: $n=1$

$$\sum_{j=0}^{1} 2^j = 1 + 2 = 3 \quad \text{and} \quad 2^{1+1} - 1 = 4 - 1 = 3$$

Induction hypothesis;

Assume $\sum_{j=0}^{k} 2^j = 2^{k+1} - 1$

Then $\sum_{j=0}^{k+1} 2^j = \sum_{j=0}^{k} 2^j + 2^{k+1}$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

Since $P(n=k) \Rightarrow P(n=k+1)$, the statement holds and $P(n=1)$ for all $n \geq 1$.

Problem 4 : (8 points) Find the remainder when the sum

$$\sum_{i=1}^{100} i^5 = 1^5 + 2^5 + \cdots + 100^5$$

is divided by 5.

We have: $1^5 \equiv 1 \mod 5$

$2^5 \equiv 4 \cdot 4 \cdot 2 \equiv (-1)(-1)2 \equiv 2 \mod 5$

$3^5 \equiv 9 \cdot 9 \cdot 3 \equiv (-1)(-1)3 \equiv 3 \mod 5$

$4^5 \equiv (-1)^5 \equiv -1 \equiv 4 \mod 5$

$0^5 \equiv 0 \mod 5$

$$\sum_{i=1}^{100} i^5 = \sum_{n=0}^{19} \left((5n+1)^5 + (5n+2)^5 + (5n+3)^5 + (5n+4)^5 + (5n+5)^5 \right)$$

$$\equiv \sum_{n=0}^{19} (1^5 + 2^5 + 3^5 + 4^5 + 0^5) \mod 5$$

$$\equiv \sum_{n=0}^{19} (1+2+3+4+0) \mod 5$$

$$\equiv \sum_{n=0}^{19} 0 \equiv 0 \mod 5$$

so the remainder is 0.

Problem 5 : (4 points)
For each question you may justify your answer with a theorem from class or a multiplication table.

a) (2 points) List all units in the ring $\mathbb{Z}/8\mathbb{Z}$.

In $\mathbb{Z}/n\mathbb{Z}$, a is a unit if $\gcd(a,n)=1$.

The integers $0 \leq a \leq 7$ such that $\gcd(a,8)=1$ are:

$1, 3, 5, 7$

These are the units in $\mathbb{Z}/8\mathbb{Z}$.

b) (2 points) List all zero divisors in the ring $\mathbb{Z}/8\mathbb{Z}$.

In $\mathbb{Z}/n\mathbb{Z}$, a is a zero divisor if $\gcd(a,n)>1$.

The integers $0 \leq a \leq 7$ such that $\gcd(a,8)>1$ are

$0, 2, 4, 6$

These are the zero divisors in $\mathbb{Z}/8\mathbb{Z}$

Problem 6 : (8 points)

a) (5 points) Find all **integer** solutions to the equation

$$17x + 16y = 5.$$

$\gcd(17,16) = 1$ and $17 \cdot 1 + 16(-1) = 5$

Therefore $17 \cdot 5 + 16(-5) = 5$ and a particular solution is $x_p = 5$, $y_p = -5$.

Therefore all integer solutions are given by

$$x = x_p + \frac{b}{\gcd(a,b)} t = 5 + 16t$$

$$y = y_p - \frac{a}{\gcd(a,b)} t = -5 - 17t \qquad t \in \mathbb{Z}$$

b) (3 points) Find all **positive** integer solutions to the equation above.

$x > 0 \Rightarrow 5 + 16t > 0$

$16t > -5$

$t > \frac{-5}{16}$

$\Rightarrow t \geq 0$

$y > 0 \Rightarrow -5 - 17t > 0$

$-5 > 17t$

$\frac{-5}{17} > t$

$\Rightarrow t \leq -1$

There is no value of t satisfying both inequalities and therefore there are no positive integer solutions.

Problem 7 : (8 points) For a an arbitrary integer, show that

$$\gcd(2a+1, 9a+4) = 1.$$

By a theorem from class, it suffices to find $x, y \in \mathbb{Z}$ such that

$$(2a+1)x + (9a+4)y = 1$$

$$2ax + x + 9ay + 4y = 1$$

$$(2x+9y)a + (x+4y) = 1$$

So we solve $\begin{matrix} 2x+9y=0 \\ x+4y=1 \end{matrix} \sim \begin{matrix} 2x+9y=0 \\ \underline{2x+8y=2} \\ y=-2 \end{matrix}$

then $x = 1-4y = 1-4(-2) = 9$

We have that for any $a \in \mathbb{Z}$,

$$(2a+1)9 + (9a+4)(-2) = 1$$

so $\gcd(2a+1, 9a+4) = 1$ since $9, -2 \in \mathbb{Z}$