Math 255: Spring 2018
Exam 1

NAME:

Time: **50 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Problem	Value	Score
1	4	
2	4	
3	4	
4	6	
5	5	
6	6	
7	8	
8	6	
9	7	
GC	5	
TOTAL	50 (or 55)	

Problem 1 : (4 points) Compute $(143, 227)$. You may use any technique you like, but you must justify your answer to receive credit.

Problem 2 : (4 points) Compute 13^{-1} modulo 15.

Problem 3 : (4 points) Of the equations below, circle all of the ones that have integer solutions.

You do not need to justify your answer to receive credit, but your justification will be taken into account to award partial credit if necessary.

a) $4x + 6y = 5$

b) $2x + 3y = 7$

c) $7x + 14y = 1$

d) $16x + 28y = 4$

Problem 4 : (6 points) Give all integer solutions to

$$6x + 14y = 6.$$

If there are no solutions, please state "None."

Problem 5 : (5 points)

a) (3 points) Let a, b and m be integers, with $m > 1$. Give the definition of the expression

$$a \equiv b \pmod{m}.$$

b) (2 points) Let a and m be integers, with $m > 1$. Show that m divides a if and only if $a \equiv 0 \pmod{m}$.

Problem 6 : (6 points) It is a theorem that:

Every integer a is congruent $\pmod{m}$ to exactly one of $0, 1, \ldots, m-2, m-1$.

Furthermore, we call this integer the *least residue of* $a \pmod{m}$.
Perform each of the following operations, and **give your answer as a least residue.** For example, the answer to

$$3 \times 4 \pmod{8}$$

should be 4 (and not 12, although those are congruent modulo 8, since 12 is not a least residue modulo 8).

a) $8 + 9 \pmod{12}$

b) $-5 - 7 \pmod{9}$

c) $8 \times 6 \pmod{11}$

Problem 7 : (8 points)

a) (4 points) Let n be an **odd** integer (this means that there is an integer k such that $n = 2k + 1$).
   List all of the possible **least residues** for $n \pmod 8$.

b) (4 points) If n is odd, list all of the possible **least residues** for $n^2 \pmod 8$.

Problem 8 : (6 points) Use induction on n to show that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for each integer $n \geq 1$.

Problem 9 : (7 points)

a) (3 points) Give the definition of *prime*.

b) (4 points) How many primes are there that have the last digit 5? Justify your answer.

Extra problem for graduate credit:

Problem 10 : (5 points) In this question, you will show that:

If a and b are integers and $(a, b) = 1$, then $(a + b, a - b) = 1$ or 2.

a) (2 points) Let $a, b \in \mathbb{Z}$. Show that $(a + b, a - b) \leq (2a, 2b)$.

b) (2 points) Let $a, b \in \mathbb{Z}$. Show that if $(a, b) = 1$, then $(2a, 2b) = 2$.

c) (1 point) Assuming parts a) and b) (even if you did not prove them), conclude that if $a, b \in \mathbb{Z}$ and $(a, b) = 1$, then $(a + b, a - b) = 1$ or 2.