Math 255: Spring 2017
Exam 1

NAME:

Time: **50 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Problem	Value	Score
1	8	
2	6	
3	6	
4	6	
5	6	
6	6	
7	12	
TOTAL	50	

Problem 1 : (8 points)

a) (4 points) Give the definition of the word “unit,” in the context of this class.

b) (4 points) Compute 7^{-1} modulo 23.

Problem 2 : (6 points) Give all integer solutions to

$$24x + 40y = 16.$$

If there are no solutions, please state "None."

Problem 3 : (6 points) Give all solutions of this equation in the ring $\mathbb{Z}/102\mathbb{Z}$:

$$36x \equiv 8 \pmod{102}.$$

If there are no solutions, please state "None."

Problem 4 : (6 points) Show that the fourth power of any integer is either of the form $5k$ or $5k+1$ for $k \in \mathbb{Z}$.

Problem 5 : (6 points) If $r \neq 1$, show that for every $n \geq 0$,

$$a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}.$$

Problem 6 : (6 points) Show that a divides b if and only if ac divides bc for all $c \neq 0$.

Problem 7 : (12 points)

a) (4 points) Give all solutions of this equation in the ring $\mathbb{Z}/4\mathbb{Z}$:

$$2x \equiv 2 \pmod 4$$

b) (6 points) Consider now the following set of simultaneous congruences:

$$\begin{aligned} 2x &\equiv 1 \pmod 3 \\ 2x &\equiv 2 \pmod 4 \\ 2x &\equiv 1 \pmod 5 \end{aligned}$$

Note that the middle congruence is the one you solved in part a).

There is an integer n such that there are exactly two solutions to this set of congruences in $\mathbb{Z}/n\mathbb{Z}$. Give this value of n and the two solutions.
Hint: Do the Chinese Remainder Theorem with each of the solutions for part a).

This problem is continued on the next page.

For your convenience, the set of congruences is

$$\begin{aligned} 2x &\equiv 1 \pmod{3} \\ 2x &\equiv 2 \pmod{4} \\ 2x &\equiv 1 \pmod{5} \end{aligned}$$

c) (2 points) What is the smallest positive integer that is a solution of this set of congruences?