Math 255 - Spring 2017
Solutions to suggested problems
Problems between April 14 and May 5 (Final Exam)

Please note: If there are any typos, please post about them on Piazza. The latest corrections to the solutions will be available there.

Section 9.1

1. (a) We use the quadratic formula:

$$x \equiv \frac{-7 + \text{``}\sqrt{49-40}\text{''}}{2} \equiv \frac{-7 + \text{``}\sqrt{9}\text{''}}{2} \equiv \frac{-7 \pm 3}{2} \pmod{11}.$$

This is $x \equiv -5, -2 \pmod{11}$ or $x \equiv 6, 9 \pmod{11}$. Here we use the fact that we know a square root of 9 modulo any n, and since 11 is an odd prime, $y^2 \equiv 9 \pmod{11}$ has two solutions x_1 and $-x_1$.

(b) Again we use the quadratic formula:

$$x \equiv \frac{-9 + \text{``}\sqrt{81-84}\text{''}}{6} \equiv \frac{-9 + \text{``}\sqrt{-3}\text{''}}{6} \pmod{13}.$$

Therefore everything hangs on whether $y^2 \equiv -3 \equiv 10 \pmod{13}$ has a solution and what it is. Example 9.1 tackles exactly this problem and gives us the solutions $y \equiv 6, 7 \pmod{13}$. (We could have also computed all of the squares modulo 13 ourselves; 13 is not too big.) We will also need to know what $6^{-1} \pmod{13}$ is. Since $2 \cdot 6 = 12 \equiv -1 \pmod{13}$, $6^{-1} \equiv -2 \equiv 11 \pmod{13}$. Now we plug all of this into the quadratic equation to get

$$x \equiv \frac{-9+6}{6} \equiv 11 \cdot (-3) \equiv -33 \equiv 6 \pmod{13}$$

and

$$x \equiv \frac{-9+7}{6} \equiv 11 \cdot (-2) \equiv -22 \equiv 4 \pmod{13}$$

(c) We use the quadratic formula once more:

$$x \equiv \frac{-6 + \text{``}\sqrt{36-20}\text{''}}{10} \equiv \frac{-6 + \text{``}\sqrt{16}\text{''}}{10} \pmod{23}.$$

Since 23 is an odd prime, we know that $y^2 \equiv 16 \pmod{23}$ has two solutions x_1 and $-x_1$. We know a solution since 16 is a square in the integers, therefore both solutions are $y \equiv 4, -4 \equiv 4, 19 \pmod{23}$. Now to complete the problem we need to find $10^{-1} \pmod{23}$. We use the Euclidean algorithm

$$23 = 2 \cdot 10 + 3$$
$$10 = 3 \cdot 3 + 1,$$

then backsolve

$$1 = 10 - 3 \cdot 3 = 10 - 3(23 - 2 \cdot 10) = 10 - 3 \cdot 23 + 6 \cdot 10 = 7 \cdot 10 - 3 \cdot 23,$$

and $10^{-1} \equiv 7 \pmod{23}$. Therefore the solutions are

$$x \equiv \frac{-6+4}{10} \equiv (-2) \cdot 7 \equiv -14 \equiv 9 \pmod{23}$$

and

$$x \equiv \frac{-6+19}{10} \equiv 13 \cdot 7 \equiv 91 \equiv 22 \pmod{23}.$$

4. We compute $\left(\frac{3}{23}\right)$ and $\left(\frac{3}{31}\right)$. There are many ways to do this, and this section probably expects us to use Euler's Criterion, but now that we know Quadratic Reciprocity, since both 23 and 31 are primes, we might as well use it since it's faster. We have

$$\left(\frac{3}{23}\right) = (-1)^{\frac{3-1}{2}\frac{23-1}{2}} \left(\frac{23}{3}\right) = -\left(\frac{2}{3}\right) = 1.$$

Therefore 3 is indeed a quadratic residue modulo 23. Now onto 31:

$$\left(\frac{3}{31}\right) = (-1)^{\frac{3-1}{2}\frac{31-1}{2}} \left(\frac{31}{3}\right) = -\left(\frac{1}{3}\right) = -1,$$

and 3 is a quadratic nonresidue modulo 31.

11. (a) By Theorem 9.1, any even power of a primitive root is a quadratic residue. Therefore the quadratic residues modulo 19 are

$$2^2 \equiv 4 \pmod{19}, \quad 2^4 \equiv 16 \pmod{19}, \quad 2^6 \equiv 64 \equiv 7 \pmod{19},$$
$$2^8 \equiv 28 \equiv 9 \pmod{19}, \quad 2^{10} \equiv 36 \equiv 17 \pmod{19}, \quad 2^{12} \equiv 68 \equiv 11 \pmod{19}$$
$$2^{14} \equiv 44 \equiv 6 \pmod{19}, \quad 2^{16} \equiv 24 \equiv 5 \pmod{19}, \quad 2^{18} \equiv 20 \equiv 1 \pmod{19}.$$

Section 9.2

1. (a) Since both 19 and 23 are odd primes we can use Quadratic Reciprocity:

$$\left(\frac{19}{23}\right) = (-1)^{\frac{19-1}{2}\frac{23-1}{2}} \left(\frac{23}{19}\right) = -\left(\frac{4}{19}\right) = -1$$

(b) We note that both 23 and 59 are odd primes so we can use Quadratic Reciprocity:

$$\begin{aligned}\left(\frac{-23}{59}\right) &= \left(\frac{-1}{59}\right)\left(\frac{23}{59}\right) = (-1)(-1)^{\frac{23-1}{2}\frac{59-1}{2}} \left(\frac{59}{23}\right) \\ &= \left(\frac{13}{23}\right) = (-1)^{\frac{13-1}{2}\frac{23-1}{2}} \left(\frac{23}{13}\right) = \left(\frac{10}{13}\right) = 1\end{aligned}$$

Here we used that $53 \equiv 3 \pmod{4}$ so $\left(\frac{-1}{59}\right) = -1$, that 13 is also an odd prime, and the results of Example 9.1.

(c) We cannot use Quadratic Reciprocity right away because 20 is not a prime, although we will later:

$$\left(\frac{20}{31}\right) = \left(\frac{4}{31}\right)\left(\frac{5}{31}\right) = \left(\frac{5}{31}\right) = (-1)^{\frac{5-1}{2}\frac{31-1}{2}}\left(\frac{31}{5}\right) = \left(\frac{1}{5}\right) = 1.$$

(d) Again, 18 is not prime, so we cannot use Quadratic Reciprocity.

$$\left(\frac{18}{43}\right) = \left(\frac{2}{43}\right)\left(\frac{9}{43}\right) = -1.$$

Here we have used that $43 \equiv 3 \pmod 8$ so $\left(\frac{2}{43}\right) = -1$.

(e) Again we cannot use Quadratic Reciprocity. I belabor this point because this is a common mistake. Quadratic Reciprocity is only for odd primes.

$$\left(\frac{-72}{131}\right) = \left(\frac{-1}{131}\right)\left(\frac{2}{131}\right)\left(\frac{36}{131}\right) = (-1)(-1) = 1.$$

Section 9.3

1. (a) Both 71 and 73 are odd primes, so we use Quadratic Reciprocity

$$\left(\frac{71}{73}\right) = (-1)^{\frac{71-1}{2}\frac{73-1}{2}}\left(\frac{73}{71}\right) = \left(\frac{2}{71}\right) = 1$$

(b) See solutions to Quiz 24.

(c) Again 461 and 773 are odd primes:

$$\begin{aligned}\left(\frac{461}{773}\right) &= (-1)^{\frac{461-1}{2}\frac{773-1}{2}}\left(\frac{773}{461}\right) = \left(\frac{312}{461}\right) = \left(\frac{2}{461}\right)\left(\frac{4}{461}\right)\left(\frac{3}{461}\right)\left(\frac{13}{461}\right)\\ &= -(-1)^{\frac{3-1}{2}\frac{461-1}{2}}\left(\frac{461}{3}\right)(-1)^{\frac{13-1}{2}\frac{461-1}{2}}\left(\frac{461}{13}\right) = \left(\frac{2}{3}\right)\left(\frac{6}{13}\right)\\ &= -\left(\frac{2}{13}\right)\left(\frac{3}{13}\right) = -(-1)^{\frac{3-1}{2}\frac{13-1}{2}}\left(\frac{13}{3}\right) = -\left(\frac{1}{3}\right) = -1\end{aligned}$$

(d) These are getting too ridiculous for the final but they are fun:

$$\begin{aligned}\left(\frac{1234}{4567}\right) &= \left(\frac{2}{4567}\right)\left(\frac{617}{4567}\right) = (-1)^{\frac{617-1}{2}\frac{4567-1}{2}}\left(\frac{4567}{617}\right) = \left(\frac{248}{617}\right)\\ &= \left(\frac{2}{617}\right)\left(\frac{4}{617}\right)\left(\frac{31}{617}\right) = (-1)^{\frac{31-1}{2}\frac{617-1}{2}}\left(\frac{617}{31}\right) = \left(\frac{28}{31}\right)\\ &= \left(\frac{4}{31}\right)\left(\frac{7}{31}\right) = (-1)^{\frac{7-1}{2}\frac{31-1}{2}}\left(\frac{31}{7}\right) = -\left(\frac{3}{7}\right) = -(-1)^{\frac{3-1}{2}\frac{7-1}{2}}\left(\frac{7}{3}\right)\\ &= \left(\frac{1}{3}\right) = 1\end{aligned}$$

(e)

$$\begin{aligned}\left(\frac{3658}{12703}\right) &= \left(\frac{2}{12703}\right)\left(\frac{31}{12703}\right)\left(\frac{59}{12703}\right)\\ &= (-1)^{\frac{31-1}{2}\frac{12703-1}{2}}\left(\frac{12703}{31}\right)(-1)^{\frac{59-1}{2}\frac{12703-1}{2}}\left(\frac{12703}{59}\right)\\ &= \left(\frac{24}{31}\right)\left(\frac{18}{59}\right) = \left(\frac{2}{31}\right)\left(\frac{3}{31}\right)\left(\frac{4}{31}\right)\left(\frac{2}{59}\right)\left(\frac{9}{59}\right)\\ &= (-1)^{\frac{3-1}{2}\frac{31-1}{2}}\left(\frac{31}{3}\right)(-1) = \left(\frac{1}{3}\right) = 1\end{aligned}$$

3. (a) This amounts to computing $\left(\frac{219}{419}\right)$, since 419 is an odd prime.

$$\left(\frac{219}{419}\right) = (-1)^{\frac{219-1}{2}\frac{419-1}{2}}\left(\frac{419}{219}\right) = -\left(\frac{200}{219}\right) = -\left(\frac{2}{219}\right)\left(\frac{100}{219}\right) = 1.$$

Since the Legendre symbol is 1, the equation is solvable.

(b) To solve this equation we would use the quadratic formula, which would ask us to compute the solutions to $y^2 \equiv b^2 - 4ac \pmod{89}$. If this has solutions, then the more complicated equation will have solutions. Here we have

$$b^2 - 4ac \equiv 6^2 - 4 \cdot 3 \cdot 5 \equiv 36 - 60 \equiv -24 \pmod{89}.$$

Therefore we are interested in the value of

$$\left(\frac{-24}{89}\right) = \left(\frac{-1}{89}\right)\left(\frac{2}{89}\right)\left(\frac{3}{89}\right)\left(\frac{4}{89}\right) = (-1)^{\frac{3-1}{2}\frac{89-1}{2}}\left(\frac{89}{3}\right) = \left(\frac{2}{89}\right) = -1.$$

Since there is no square root, the quadratic equation is not solvable.

(c) Similarly to part (b), we are interested in figuring out if

$$b^2 - 4ac \equiv 5^2 - 4 \cdot 2 \cdot (-9) \equiv 25 + 72 \equiv 97 \equiv -4 \pmod{101}$$

is a square modulo 101. So we compute

$$\left(\frac{-4}{101}\right) = \left(\frac{-1}{101}\right)\left(\frac{4}{101}\right) = 1,$$

since $101 \equiv 1 \pmod 4$. Therefore the equation is solvable.

Section 9.4

1. (a) By Theorem 9.11, the class version, the congruence $x^2 \equiv -1 \pmod{25}$ has either no solution or two solutions, since 25 is a power of an odd prime and $\gcd(-1, 25) =$

1. It has two solutions if $x^2 \equiv -1 \pmod 5$ has a solution. Since this is the equation $x^2 \equiv 4 \pmod 5$, which has solution $x \equiv 2 \pmod 5$, we conclude that $x^2 \equiv -1 \pmod{25}$ has exactly 2 solutions. It is now a simple matter to verify that the two given solutions are solutions: If $x \equiv 7 \pmod{25}$, then indeed $x^2 = 49 \equiv -1 \pmod{25}$. Also, if $x \equiv 18 \equiv -7 \pmod{25}$, then $x^2 \equiv (-7)^2 \equiv 49 \equiv -1 \pmod{25}$.

(b) We lift $x_0 = 7$ to $x_1 = 7 + 25y_0$, where $x_1^2 \equiv -1 \pmod{125}$. We have

$$x_1^2 = (7 + 25y_0)^2 = 49 + 350y_0 + 675y_0^2 \equiv 49 + 100y_0 \pmod{125}.$$

Therefore we must solve

$$-1 \equiv 49 + 100y_0 \pmod{125}$$

or

$$-50 \equiv 100y_0 \pmod{125}.$$

100 is not a unit modulo 125, but $\gcd(100, 125) = 25$ divides -50, so we may divide all the way through by 25 to solve instead

$$-2 \equiv 4y_0 \pmod 5$$

or, since $4 \equiv -1 \pmod 5$,

$$-y_0 \equiv -2 \pmod 5,$$

which has solution $y_0 \equiv 2 \pmod 5$ since -1 is a unit modulo 5. Therefore our lift is $x_1 = 7 + 25 \cdot 2 = 57$.

Since 125 is a power of an odd prime, the quadratic congruence $x^2 \equiv -1 \pmod{125}$ has two solutions. One of them is $x \equiv 57 \pmod{125}$ and the other is $x \equiv -57 \equiv 68 \pmod{125}$. Therefore the two solutions are

$$x \equiv 57 \pmod{125} \quad \text{and} \quad x \equiv 68 \pmod{125}.$$

2. (a) See the solutions to Quiz 25.

(b) We first solve $x^2 \equiv 14 \equiv 4 \pmod 5$. This has solution $x \equiv 2 \pmod 5$.

Our first lifting step is to take $x_0 = 2$ and lift it to $x_1 = 2 + 5y_0$, where $x_1^2 \equiv 14 \pmod{25}$. Squaring, we get

$$x_1^2 = (2 + 5y_0)^2 = 4 + 20y_0 + 25y_0^2 \equiv 4 + 20y_0 \pmod{25}.$$

Therefore we must solve

$$14 \equiv 4 + 20y_0 \pmod{25}$$

or

$$10 \equiv 20y_0 \pmod{25}.$$

Dividing through by $\gcd(25, 20) = 5$ (since 20 is not a unit we may not divide by 20), we get the equation

$$2 \equiv 4y_0 \pmod{5},$$

and since $4 \equiv -1 \pmod{5}$, this has solution $y_0 \equiv -2 \equiv 3 \pmod{5}$. Therefore our lift is $x_1 = 2 + 5 \cdot 3 = 17 \pmod{25}$.

Our second lifting step is to take $x_0 = 17$ and lift it to $x_1 = 17 + 25y_0$, where $x_1^2 \equiv 14 \pmod{125}$. Squaring, we get

$$x_1^2 = (17 + 25y_0)^2 = 289 + 850y_0 + 625y_0^2 \equiv 39 + 100y_0 \pmod{125}.$$

Therefore we must solve

$$14 \equiv 39 + 100y_0 \pmod{125}$$

or

$$-25 \equiv 100y_0 \pmod{125}.$$

This time $\gcd(100, 125) = 25$, so we divide all the way though by 25 to get

$$-1 \equiv 4y_0 \pmod{5}.$$

Since $4 \equiv -1 \pmod{5}$, this has solution $y_0 \equiv 1 \pmod{5}$, and the lift is $x_1 = 17 + 25 = 42$.

Therefore the two solutions to this quadratic congruence are

$$x \equiv 42 \pmod{125} \quad \text{and} \quad x \equiv -42 \equiv 83 \pmod{125}.$$

(c) We begin by solving $x^2 \equiv 2 \pmod{7}$. This can be done by exhausting all possibilities for x: If $x \equiv 1 \pmod{7}$, then $x^2 \equiv 1 \pmod{7}$; if $x \equiv 2 \pmod{7}$, then $x^2 \equiv 4 \pmod{7}$; if $x \equiv 3 \pmod{7}$, then $x^2 \equiv 2 \pmod{7}$. Therefore $x \equiv 3 \pmod{7}$ is a solution.

Our first lifting step is to lift $x_0 = 3$ to $x_1 = 3 + 7y_0$, where $x_1^2 \equiv 2 \pmod{49}$. Squaring, we get

$$x_1^2 = (3 + 7y_0)^2 = 9 + 42y_0 + 49y_0^2 \equiv 9 + 42y_0 \pmod{49}.$$

Therefore we must solve

$$2 \equiv 9 + 42y_0 \pmod{49}$$

or

$$-7 \equiv 42y_0 \pmod{49}.$$

Since $\gcd(42, 49) = 7$, we divide all the way through by 7 to get

$$-1 \equiv 6y_0 \pmod 7.$$

Since $6 \equiv -1 \pmod 7$, and -1 is a unit modulo 7, this has solution

$$y_0 \equiv 1 \pmod 7.$$

Therefore our lift is $x_1 = 3+7\cdot 1 = 10$. Note that indeed $10^2 = 100 \equiv 2 \pmod{49}$. Our second lifting is to take $x_1 = 10$ and lift it to $x_1 = 10 + 49y_0$, where $x_1^2 \equiv 2 \pmod{343}$. Squaring, we get

$$x_1^2 = (10 + 49y_0)^2 = 100 + 980y_0 + 7^4y_0^2 \equiv 100 + 294y_0 \pmod{343}.$$

Therefore we must solve

$$2 \equiv 100 + 294y_0 \pmod{343}$$

or

$$-98 \equiv 294y_0 \pmod{343}.$$

Since $294 = 7 \cdot 42 = 7 \cdot 7 \cdot 6$, $\gcd(294, 343) = 49$ and we may divide all the way through by 49 to get

$$-2 \equiv 6y_0 \pmod 7.$$

Again, since $6 \equiv -1 \pmod 7$ and -1 is a unit modulo 7, we get the solution

$$y_0 \equiv 2 \pmod 7.$$

Therefore our lift is $x_1 = 10 + 49 \cdot 2 = 10 + 98 = 108$.
The two solutions to this quadratic congruence are thus

$$x \equiv 108 \pmod{343} \quad \text{and} \quad x \equiv -108 \equiv 235 \pmod{343}.$$

9. (a) We first consider the equation

$$x^2 \equiv 3 \pmod{11^2 \cdot 23^2}.$$

To solve this equation, we would solve

$$x^2 \equiv 3 \pmod{11^2} \quad \text{and} \quad x^2 \equiv 3 \pmod{23^2}.$$

Since these are both odd prime powers and the equations are of the form $x^2 \equiv a \pmod{p^k}$ with $\gcd(a, p) = 1$, each equation has either 0 or 2 solutions. The first equation has two solutions if and only if $\left(\frac{3}{11}\right) = 1$, so we compute this symbol:

$$\left(\frac{3}{11}\right) = (-1)^{\frac{3-1}{2}\frac{11-1}{2}} \left(\frac{11}{3}\right) = -\left(\frac{2}{3}\right) = 1.$$

The second equation has two solutions if and only if $\left(\frac{3}{23}\right)$, so now compute this new symbol:

$$\left(\frac{3}{23}\right) = (-1)^{\frac{3-1}{2}\frac{23-1}{2}}\left(\frac{23}{3}\right) = -\left(\frac{2}{3}\right) = 1.$$

This second equation has two solutions, therefore the equation $x^2 \equiv 3 \pmod{11^2 \cdot 23^2}$ has $2 \cdot 2 = 4$ solutions.

We now consider the equation

$$x^2 \equiv 9 \pmod{2^3 \cdot 3 \cdot 5^2}.$$

This time we would need to solve

$$x^2 \equiv 9 \pmod{2^3}, \quad x^2 \equiv 9 \pmod{3} \quad \text{and} \quad x^2 \equiv 9 \pmod{5^2}.$$

The first equation, $x^2 \equiv 9 \pmod{2^3}$, has 4 solutions by Theorem 9.12 since $9 \equiv 1 \pmod{8}$.

The second equation, $x^2 \equiv 9 \equiv 0 \pmod{3}$ has the unique solution $x \equiv 0 \pmod{3}$.

The third equation, $x^2 \equiv 9 \pmod{25}$ has either 0 or 2 solutions, but since we see $x \equiv 3 \pmod{25}$ is a solution, the equation must have 2 solutions.

Therefore, the equation $x^2 \equiv 9 \pmod{2^3 \cdot 3 \cdot 5^2}$ has $4 \cdot 1 \cdot 2 = 8$ solutions.

We take this opportunity to note that $x \equiv 3 \pmod{2^3 \cdot 3 \cdot 5^2}$ and $x \equiv -3 \pmod{2^3 \cdot 3 \cdot 5^2}$ will be among those, but we must use the Chinese Remainder Theorem to find the others. Their relationship to 3 and -3 is not straightforward to see with it.

(b) To find the 8 solutions, we solve the equations

$$x^2 \equiv 9 \pmod{2^3}, \quad x^2 \equiv 9 \pmod{3} \quad \text{and} \quad x^2 \equiv 9 \pmod{5^2}.$$

The first equation, $x^2 \equiv 9 \equiv 1 \pmod{2^3}$, has 4 solutions as stated above. We know that one of them is $x \equiv 1 \pmod{8}$, and we now how to make the other three out of this solution: They are $x \equiv -1 \equiv 7 \pmod{8}$, $x \equiv 1+4 \equiv 5 \pmod{8}$ and $x \equiv -5 \equiv 3 \pmod{8}$.

The second equation has unique solution $x \equiv 0 \pmod{3}$, as stated above.

The third equation has two solutions, one of them is $x \equiv 3 \pmod{25}$. We know that the other is $x \equiv -3 \equiv 22 \pmod{25}$.

Now, to be explicit, the 8 solutions are the 8 solutions to these 8 Chinese Remainder problems:

$$\begin{aligned}
x \equiv 1 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 3 \pmod{25} \\
x \equiv 1 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 22 \pmod{25} \\
x \equiv 3 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 3 \pmod{25} \\
x \equiv 3 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 22 \pmod{25} \\
x \equiv 5 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 3 \pmod{25} \\
x \equiv 5 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 22 \pmod{25} \\
x \equiv 7 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 3 \pmod{25} \\
x \equiv 7 \pmod 8, \quad x \equiv 0 \pmod 3, \quad \text{and} \quad x \equiv 22 \pmod{25}
\end{aligned}$$

We note that in the notation of the Chinese Remainder Theorem, N_i and x_i do not depend on a_i. We also note that since in all cases $a_2 = 0$, we do not need N_2 and x_2. Therefore we quickly find N_1, x_1, N_3 and x_3 once and for all.
We have that $N_1 = 3 \cdot 125 = 375$ and x_1 is a solution to $375x_1 \equiv 1 \pmod 8$. Since $375 \equiv 7 \equiv -1 \pmod 8$, we can choose $x_1 = -1$.
We have that $N_3 = 8 \cdot 3 = 24$ and x_3 is a solution to $24x_3 \equiv 1 \pmod{25}$. Since $24 \equiv -1 \pmod{25}$, again we can choose $x_3 = -1$.
Therefore, for each pair (a_1, a_3) (since a_2 is always 0), the solution we seek is

$$x \equiv -375a_1 - 24a_3 \pmod{600}.$$

Going through each of the possibilities above in order, we get

$$\begin{aligned}
x &\equiv 153 \pmod{600} \\
x &\equiv 297 \pmod{600} \\
x &\equiv 3 \pmod{600} \\
x &\equiv 147 \pmod{600} \\
x &\equiv 453 \pmod{600} \\
x &\equiv 597 \pmod{600} \\
x &\equiv 303 \pmod{600} \\
x &\equiv 447 \pmod{600}.
\end{aligned}$$

We note that these form 4 pairs of solutions $(x_i, -x_i)$, with $x_1 = 3$ $(-3 \equiv 597 \pmod{600})$, $x_2 = 147$ $(-147 \equiv 453 \pmod{600})$, $x_3 = 153$ $(-153 \equiv 447 \pmod{600})$, and $x_4 = 297 \pmod{600}$ $(-297 \equiv 303 \pmod{600})$.
One way we could think of these solutions is that they are all lifts of ± 3 modulo 150. This could be because 150 is the largest factor of 600 where $x^2 \equiv 9 \pmod n$ has only two solutions. I am not sure because I would have to prove a theorem to check it, but it's possible (likely?) that the solutions always arrange themselves that way.