Math 255 - Spring 2017
Homework 7 Solutions

1. (a) Consider first $n = 1$. Then $\log(1) = 0$ and $\sum_{d|1} \Lambda(d) = \Lambda(1) = 0$, since 1 is not of the form p^k for p a prime and $k \geq 1$.

   Now let $n > 1$. By the Fundamental Theorem of Arithmetic, there are primes $p_1, \ldots, p_r$ and powers $k_1, \ldots k_r$ with each $k_i \geq 1$ such that

$$n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} = \prod_{i=1}^{r} p_i^{k_i}.$$

   By Theorem 6.1, the positive divisors of n are precisely the integers of the form

$$p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$$

   where $0 \leq a_i \leq k_i$ for $i = 1, 2, \ldots, r$. Therefore, we have

$$\begin{aligned}
\sum_{d|n} \Lambda(d) &= \sum_{\substack{d = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \\ 0 \leq a_i \leq k_i}} \Lambda(d) \\
&= \sum_{\substack{d = p_i^{k_i} \\ 1 \leq a_i \leq k_i}} \Lambda(d) \\
&= \sum_{j=1}^{k_1} \Lambda(p_1^j) + \sum_{j=1}^{k_2} \Lambda(p_2^j) + \cdots + \sum_{j=1}^{k_r} \Lambda(p_r^j) \\
&= \sum_{j=1}^{k_1} \log(p_1) + \sum_{j=1}^{k_2} \log(p_2) + \cdots + \sum_{j=1}^{k_r} \log(p_r) \\
&= k_1 \log(p_1) + k_2 \log(p_2) + \cdots + k_r \log(p_r) \\
&= \log(p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}) = \log(n).
\end{aligned}$$

   Above the second equality is because $\Lambda(d)$ is zero on the integers d that are not of the form in the second sum, and the second to last equality is by a law of logarithms.

   (b) By Möbius Inversion (we note that Λ is not multiplicative, but Möbius Inversion applies to any number-theoretic function) we have that

$$\Lambda(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log d = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right).$$

   The first equality takes cares of the first equality we must prove.

For the second equality we have

$$\begin{aligned}
\Lambda(n) &= \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) \\
&= \sum_{d|n} \mu(d) \left(\log(n) - \log(d)\right) \\
&= \sum_{d|n} \mu(d) \log(n) - \sum_{d|n} \mu(d) \log(d) \\
&= \log(n) \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log(d).
\end{aligned}$$

By Theorem 6.6, we have that $\sum_{d|n} \mu(d) = 0$ if $n \neq 1$ and $\sum_{d|n} \mu(d)$ if $n = 1$. We therefore consider the two cases separately. If $n = 1$, then $\log(1) = 0$, so

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log(d).$$

In the case where $n \neq 1$, the first sum is 0 still, so again we have

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log(d).$$

2. (a) We have that

$$\mu(d) = \begin{cases} 0 & \text{if } d \text{ is not square-free,} \\ \pm 1 & \text{if } d \text{ is square-free,} \end{cases}$$

if we take $d = 1$ to be square-free. Therefore,

$$|\mu(d)| = \begin{cases} 0 & \text{if } d \text{ is not square-free,} \\ 1 & \text{if } d \text{ is square-free.} \end{cases}$$

As a consequence, the function

$$\sum_{d|n} |\mu(d)|$$

exactly counts the square-free divisors of n, and therefore is the function $S(n)$.

(b) We know that μ is a multiplicative function (Theorem 6.5). Now let m, n be positive integers with $\gcd(m, n) = 1$. Then

$$\begin{aligned}
|\mu(mn)| &= |\mu(m)\mu(n)| \\
&= |\mu(m)||\mu(n)|,
\end{aligned}$$

so $|\mu|$ is also multiplicative. Therefore by Theorem 6.4, $S(n) = \sum_{d|n} |\mu(d)|$ is multiplicative.

(c) Let us first tackle the case of $n = 1$: On the one hand, $n = 1$ only has one positive divisor, $d = 1$, and since there is no prime with p^2 dividing 1, $S(1) = 1$. On the other hand, $\omega(1) = 0$ since no prime divides 1, so $2^{\omega(1)} = 2^0 = 1$. Therefore $S(1) = 2^{\omega(1)}$.

Since S is multiplicative, we can compute its value by first computing $S(p^k)$ for p a prime and $k \geq 1$, then patch together using multiplicativity. On the one hand, we have

$$\begin{aligned} S(p^k) &= \sum_{d \mid p^k} |\mu(d)| \\ &= \sum_{j=0}^{k} |\mu(p^j)| \\ &= |\mu(1)| + |\mu(p)| \\ &= 2, \end{aligned}$$

where we use that $|\mu(p^j)| = 0$ if $j \geq 2$, because in that case p^j is not square-free. On the other hand, $\omega(p^k) = 1$, since p is the only prime dividing p^k when $k \geq 1$. Therefore $2^{\omega(p^k)} = 2^1 = 2$, and indeed $S(p^k) = 2^{\omega(p^k)}$.

Now let $n > 1$ and write $n = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$ for p_i some primes and integers $k_i \geq 1$. Note that for such an n, $\omega(n) = r$. Using multiplicativity of S, we have

$$\begin{aligned} S(n) &= S(p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}) \\ &= S(p_1^{k_1}) S(p_2^{k_2}) \ldots S(p_r^{k_r}) \\ &= 2 \cdot 2 \cdots 2 \\ &= 2^r = 2^{\omega(n)}, \end{aligned}$$

and this completes the proof.

3. (a) If n is odd, then $\gcd(2, n) = 1$. Since φ is multiplicative and $\varphi(2) = 1$, we have

$$\varphi(2n) = \varphi(2)\varphi(n) = \varphi(n).$$

(b) Now let n be even, and write $n = 2^k m$ for some odd integer m and some integer $k \geq 1$ ($k \geq 1$ since n is divisible by 2). Then $2n = 2^{k+1}m$, $\gcd(2^{k+1}, m) = 1$, and since φ is multiplicative, we have

$$\begin{aligned} \varphi(2^{k+1}m) &= \varphi(2^{k+1})\varphi(m) \\ &= \left(2^{k+1} - 2^k\right) \varphi(m) \\ &= 2\left(2^k - 2^{k-1}\right) \varphi(m) \\ &= 2\varphi(2^k)\varphi(m) \\ &= 2\varphi(2^k m) = 2\varphi(n). \end{aligned}$$