Math 255 - Spring 2017
Homework 2 Solutions

1. (a) Let a be any integer. Then by the Division Algorithm, $a = 3q + r$ with $r = 0, 1$ or 2. We have

$$\begin{aligned} a^2 &= (3q + r)^2 \\ &= 9q^2 + 6qr + r^2 \\ &= 3(3q^2 + 2qr) + r^2. \end{aligned}$$

Let $\ell = 3q^2 + 2qr \in \mathbb{Z}$. Then

$$a^2 = 3\ell + r^2.$$

If $r = 0$, we are done: it suffices to let $k = \ell$ and $a^2 = 3k$.
If $r = 1$, again we are done: it suffices to let $k = \ell$ and $a^2 = 3k + 1$.
Finally, if $r = 2$, then $r^2 = 4 = 3+1$. If we let $k = \ell+1$, then $a^2 = 3(\ell+1)+1 = 3k + 1$.

(b) Let a be any integer. Then by the Division Algorithm, $a = 4q+r$ with $r = 0, 1, 2$ or 3. We have

$$\begin{aligned} a^2 &= (4q + r)^2 \\ &= 16q^2 + 8qr + r^2 \\ &= 4(4q^2 + 2qr) + r^2. \end{aligned}$$

Let $\ell = 4q^2 + 2qr \in \mathbb{Z}$. Then

$$a^2 = 4\ell + r^2.$$

If $r = 0$, we are done: it suffices to let $k = \ell$ and $a^2 = 4k$.
If $r = 1$, again we are done: it suffices to let $k = \ell$ and $a^2 = 4k + 1$.
If $r = 2$, then $r^2 = 4$. If we let $k = \ell + 1$, then $a^2 = 4(\ell + 1) = 4k$.
Finally, if $r = 3$, then $r^2 = 9 = 2 \times 4 + 1$. If we let $k = \ell + 2$, then $a^2 = 4(\ell + 2) + 1 = 4k + 1$.

2. We begin by tackling the base case: If $n = 1$, then $2^{4n} - 1 = 16 - 1 = 15$. Since $15 = 15 \cdot 1$, $15|15$ and the claim is proved.

   Now assume that for some $k \geq 1$, $15|(2^{4k} - 1)$. We prove that it follows that $15|(2^{4(k+1)} - 1)$.

   By the induction hypothesis, we have

$$2^{4k} - 1 = 15q$$

   for $q \in \mathbb{Z}$, or

$$2^{4k} = 15q + 1.$$

   Consider now

$$\begin{aligned} 2^{4(k+1)} - 1 &= 2^4 \cdot 2^{4k} - 1 \\ &= 16(15q + 1) - 1 \\ &= (15 + 1)(15q + 1) - 1 \\ &= 15(15q + 1) + 15q + 1 - 1 \\ &= 15(16q + 1). \end{aligned}$$

   Therefore by the definition of divisibility, $15|(2^{4(k+1)} - 1)$ and the induction step is complete.

3. Since a and b are both odd, we may write $a = 2k+1$ and $b = 2\ell+1$ for $k, \ell \in \mathbb{Z}$. We have

$$\begin{aligned} a^4 + b^4 - 2 &= (2k+1)^4 + (2\ell+1)^4 - 2 \\ &= (16k^4 + 32k^3 + 24k^2 + 8k + 1) + (16\ell^4 + 32\ell^3 + 24\ell^2 + 8k + 1) - 2 \\ &= 16(k^4 + 2k^3 + k^2 + \ell^4 + 2\ell^3 + \ell^2) + 8(k^2 + k + \ell^2 + \ell) \\ &= 16N + 8(k^2 + k + \ell^2 + \ell) \end{aligned}$$

for

$$N = k^4 + 2k^3 + k^2 + \ell^4 + 2\ell^3 + \ell^2 \in \mathbb{Z}.$$

We now show that for any $n \in \mathbb{Z}$, $n^2 + n$ is even. This will complete the proof. By the Division Algorithm, $n = 2q + r$ for $r = 0$ or 1. Then

$$\begin{aligned} n^2 + n &= (2q+r)^2 + (2q+r) \\ &= 4q^2 + 4qr + r^2 + 2q + r \\ &= 2(2q^2 + 2qr + q) + r^2 + r. \end{aligned}$$

If $r = 0$, then $r^2 + r = 0$ and $n^2 + n = 2M$ for $M = 2q^2 + 2qr + q \in \mathbb{Z}$.
If $r = 1$, then $r^2 + r = 2$ and $n^2 + n = 2M$ for $M = 2q^2 + 2qr + q + 1 \in \mathbb{Z}$.
Therefore $n^2 + n$ is even for all n as claimed.

Returning to the expression

$$a^4 + b^4 - 2 = 16N + 8(k^2 + k + \ell^2 + \ell),$$

we may write

$$k^2 + k = 2u$$

and

$$\ell^2 + \ell = 2v$$

for some $u, v \in \mathbb{Z}$. Then

$$\begin{aligned} a^4 + b^4 - 2 &= 16N + 8(k^2 + k + \ell^2 + \ell) \\ &= 16N + 8(2u + 2v) \\ &= 16(N + u + v), \end{aligned}$$

and $a^4 + b^4 - 2$ is indeed divisible by 16.