Section 6.2

A very important multiplicative function

Def Let $n \in \mathbb{Z}_{>0}$

$$\mu(n) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } p^2 \mid n \text{ some prime } p \\ (-1)^r & \text{if } n = p_1 \cdots p_r \quad p_i \text{ distinct} \end{cases}$$

Examples

$n = 10 = 2 \cdot 5 \qquad \mu(10) = (-1)^2 = 1$

$n = 2 \qquad \mu(2) = -1$

$n = 20 = 2^2 \cdot 5 \qquad \mu(20) = 0$

μ is not totally multiplicative

Theorem

μ is multiplicative

proof: Let $m, n \in \mathbb{Z}_{>0}$ $\gcd(m,n) = 1$

Case 1: $\exists$ p prime with $p^2 \mid n$ or $p^2 \mid m$

then $p^2 \mid mn$

so $\mu(mn) = 0$

$\mu(m)\mu(n) = 0$

Case 2: That doesn't happen

$m = p_1 \ldots p_r \quad n = q_1 \ldots q_s \quad p_i \neq q_j$

Then $\mu(mn) = (-1)^{r+s}$

$\mu(m) = (-1)^r$

$\mu(n) = (-1)^s$

and $(-1)^{r+s} = (-1)^r(-1)^s$

$\square$

Möbius inversion formula

Let f be a number theoretic function

$$F(n) = \sum_{d \mid n} f(d)$$

Then

$$f(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) F(d)$$

Lemma1

Let $g(n) = \sum_{d \mid n} \mu(d)$. Then

$$g(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Wed

proof: By Theorem 6.4 (last week's Big Theorem) g is multiplicative since μ is.

Note: To compute the value of a multiplicative function it is enough to know

$g(p^k)$ for p prime $k \geq 0$

Why? Then if $m = p_1^{k_1} \dots p_r^{k_r}$

$$g(m) = g(p_1^{k_1})\, g(p_2^{k_2}) \dots g(p_r^{k_r})$$

So let's compute

$$g(1) = \mu(1) = 1$$

Now let p be a prime, $k \geq 1$

$$g(p^k) = \sum_{d \mid p^k} \mu(d)$$
$$= \mu(1) + \mu(p) + \mu(p^2) + \dots + \mu(p^{k-1}) + \mu(p^k)$$
$$= 1 + (-1)^1 + 0 + 0 \dots + 0$$
$$= 1 - 1$$
$$= 0$$

Now for any $m > 1$ $m = p_1^{k_1} \dots p_r^{k_r}$

$$g(m) = g(p_1^{k_1}) \dots g(p_r^{k_r}) = 0 \qquad \square$$

proof of Möbius inversion formula

$$F(n) = \sum_{d \mid n} f(d) \quad \text{is given}$$

Want to show $\quad f(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right)$

$$\sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d \mid n} \mu(d) \left[\sum_{c \mid \frac{d}{n}} f(c) \right]$$

$$= \sum_{d \mid n} \sum_{c \mid d/n} \mu(d) f(c)$$

We want to switch the order of summation

Let (d, c) be a pair with $d \mid n$ and $c \mid {}^{d}/n$

Then $\quad n = d\underbrace{c \cdot a}_{n/d} \quad a \in \mathbb{Z}$

$$= c \underbrace{da}_{n/c}$$

So we can say $c \mid n$ and $d \mid {}^{n}/c$ instead

$$= \sum_{c \mid n} \sum_{d \mid n/c} \mu(d) f(c)$$

$$= \sum_{c \mid n} f(c) \sum_{d \mid n/c} \mu(d)$$

$$= \sum_{c|n} f(c) \quad g\left(\frac{n}{c}\right)$$

from Lemma 1

$$= \begin{cases} 1 & \text{if } \frac{n}{c} = 1 \text{ i.e. } c = n \\ 0 & \text{otherwise} \end{cases}$$

$$= f(n)$$

$\square$

Example

$$\tau(n) = \sum_{d|n} 1 \quad \text{so} \quad 1 = \sum_{d|n} \mu\left(\frac{n}{d}\right) \tau(d)$$

for all n

$n=10$ $\mu(10)\tau(1) + \mu(5)\tau(2) + \mu(2)\tau(5) + \mu(1)\tau(10)$

$= 1 \cdot 1 + (-1) \cdot 2 + (-1) \cdot 2 + 1 \cdot 4$

$= 1 - 2 - 2 + 4 = 1$

$$\sigma(n) = \sum_{d|n} d \quad \text{so} \quad n = \sum_{d|n} \mu\left(\frac{n}{d}\right) \sigma(d)$$

for all n

$n=10$ $\mu(10)\sigma(1) + \mu(5)\sigma(2) + \mu(2)\sigma(5) + \mu(1)\sigma(10)$

$= 1 \cdot 1 + (-1) \cdot 3 + (-1) \cdot 6 + (1+2+5+10)$

$= 1 - 3 - 6 + 18 = 10$