A Search Algorithm for Friendly Numbers

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Overview

Topics:

1. Friendly and solitary numbers
   - Abundant, perfect, deficient numbers
   - Sum-of-divisors function $\sigma(n)$
   - Abundancy index $I(n)$
2. Searching for friendly numbers
   - Algorithm for narrowing search
3. Main results and data

I. Friendly and Solitary Numbers

Friendly and Solitary Numbers

- Abundant, deficient, and perfect numbers
- The sum-of-divisors function
- The abundancy index
- Friendly and solitary numbers

Abundant, deficient, and perfect numbers

Definition (Abundant, deficient, perfect)

A positive integer n is:

- **Abundant** if the sum of its proper divisors is greater than itself,
- **Deficient** if the sum of its proper divisors is less than itself,
- **Perfect** if the sum of its proper divisors is equal to itself.

- 12 is abundant: $1 + 2 + 3 + 4 + 6 = 16 > 12$.
- 5 is deficient: $1 < 5$.
- 6 is perfect: $1 + 2 + 3 = 6$.

Conjecture (Open).

There exist no odd perfect numbers.

The Sum-of-Divisors Function $\sigma(n)$

Definition (Sum-of-divisors function)

The sum-of-divisors function of n, denoted $\sigma(n)$, is the sum of its positive divisors, n itself included.

Proposition

The sum-of-divisors function is ***multiplicative****, i.e.,*

$$\sigma(ab) = \sigma(a)\sigma(b)$$

for relatively prime a, b.

Multiplicativity Example

Consider $n = 36 = 4 \times 9$:
Divisors:

1	3	9
2	6	18
4	12	36

$$\begin{aligned}\sigma(36) =& 1 + 3 + 9 + 2 + 6 + 18 + 4 + 12 + 36 \\ =& 91 \\ =& (1 + 3 + 9)(1 + 2 + 4) \\ =& \sigma(9)\sigma(4).\end{aligned}$$

The Abundancy Index

Definition (Abundancy index)

The **abundancy index** of a positive integer n, denoted $I(n)$, is the ratio $\sigma(n)/n$.

Ex: $I(6) = (1 + 2 + 3 + 6)/6 = 2.$

Proposition

A positive integer n is

- ***Abundant** if $I(n) > 2$,*
- ***Deficient** if $I(n) < 2$,*
- ***Perfect** if $I(n) = 2$.*

Abundancy Index Graph

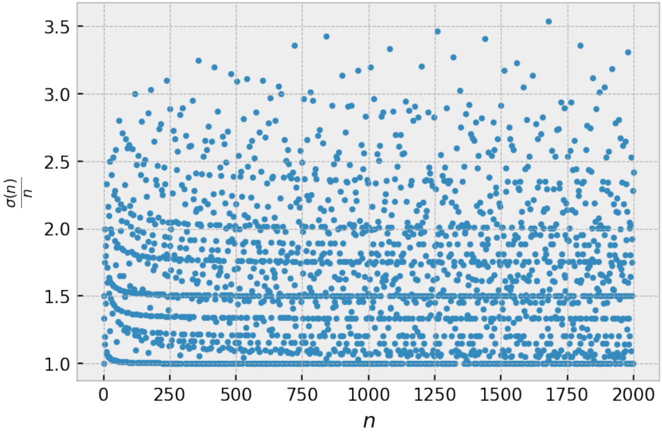

Figure: Plot of $I(n)$. J6M8, Wikimedia Commons.

Friendly and Solitary Numbers

Definition (Friendly, solitary)

A positive integer n is **friendly** if there exists another positive integer $k \neq n$ with $I(k) = I(n)$.

If no such k exists, then n is **solitary**.

- All perfect numbers are friendly, as they share $I(n) = 2$.
- Primes and their powers are solitary.
- The smallest number with unknown status is 10.

The Search Algorithm

- High-level overview
- Example run and proof: 18 is solitary

High-level overview; Useful Facts

Overview.

Given a positive integer n and large bound $B \approx 10^{30}$:

- prove that the only solution to $I(k) = I(n)$ is $k = n$ (i.e., n is solitary),
- return all positive integers $k \leq B$, $k \neq n$ with $I(k) = I(n)$, if they exist,
- search inconclusive if neither happens.

Proposition (multiplicativity)

The abundancy index $I(n)$ is multiplicative.

Proposition (divisibility ordering)

If $a \mid b$, then $I(a) \leq I(b)$ with equality only if $a = b$.

Finding Numbers with a Target Index

Example problem.

Find $n \neq 18$ with $I(n) = I(18)$, or show that no such n exists (i.e., 18 is solitary).

Observe that $I(18) = 13/6$, so we want

$$\frac{\sigma(n)}{n} = \frac{13}{6},$$

which forces $6 \mid n$. In turn, we must have

- Case (i): $2 \mid \frac{n}{6}$, i.e., $12 \mid n$
- Case (ii): $3 \mid \frac{n}{6}$, i.e., $18 \mid n$
- Case (iii): $n/6$ is relatively prime to 6.

Finding Numbers with a Target Index, cont.d

- Case (i): $12|n$. $I(12) > 13/6$.
  No multiple of 12 works due to divisibility ordering.
- Case (ii): $18|n$. $I(18) = 13/6$.
  No greater multiple of 18 works.
- Case (iii): $n/6$ is relatively prime to 6.

$$I(6) \cdot I(n/6) = I(n) = 13/6$$

$$I(n/6) = \frac{\sigma(n/6)}{n/6} = 13/12$$

  So 12 divides $n/6$, contradiction.
- Conclusion: 18 is solitary.

Speed of Algorithm

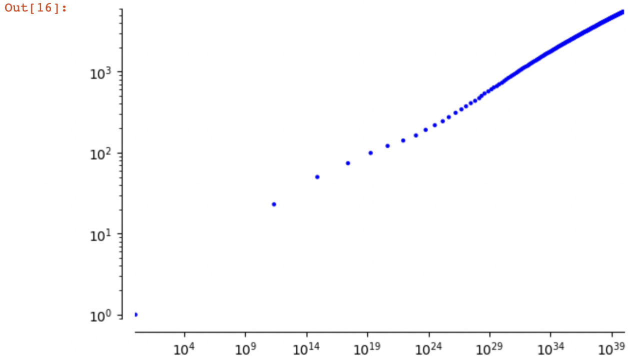

Figure: This graph plots the depth we check (horizontal) versus the number of cases required (vertical), when checking for friends of 14.

Calculations

After implementation on Sage:

- Calculated friendly status for numbers up to 3189
- Checked for friends of these numbers up to 10^{30}
- Found numbers with a small amount of cases, which heuristically are more likely to be solitary
- Friendly: 463/3189 (0.145)
  - Known to have positive natural density
- Solitary: 1376/3189 (0.431)
- Unknown: 1350/3189 (0.423)

Plot: Friendly Status of $n \leq 100$

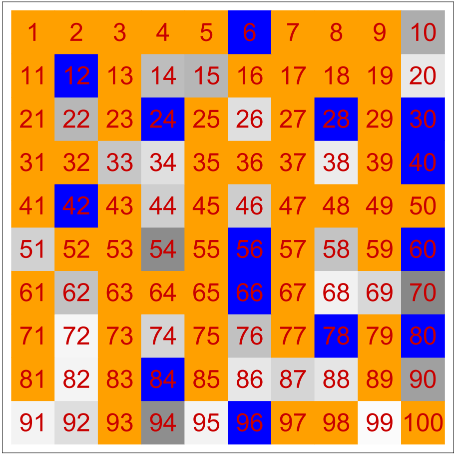

Figure: Blue confirmed friendly, Orange confirmed solitary, Gray unknown (darker shade = more cases checked up to a fixed depth).

Plot: Friendly Status of $n \leq 3000$

Organized in 30 rows of 100, with the first row being 1 through 100:

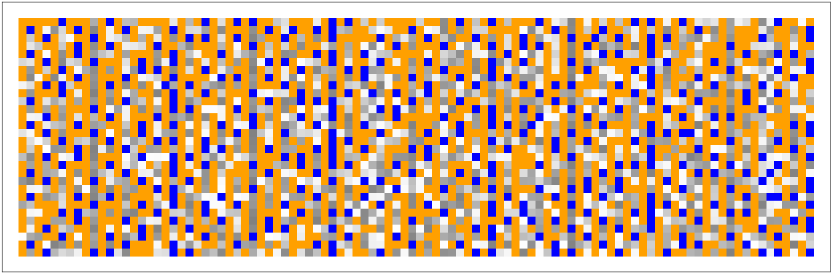

Figure: Blue confirmed friendly, Orange confirmed solitary, Gray unknown (darker shade = more cases checked up to a fixed depth).

Smallest Friends

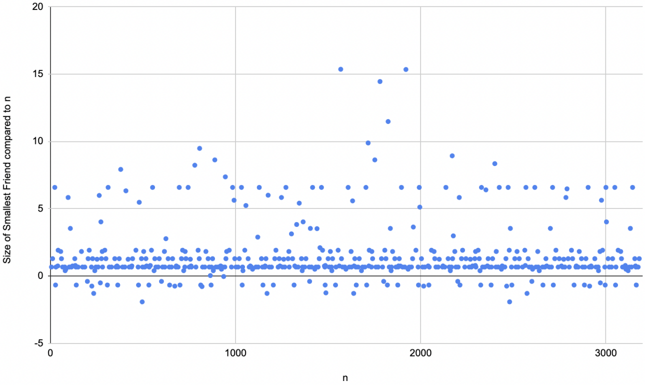

Figure: For all friendly numbers n, plot of n versus $\log(\frac{\text{smallest friend}}{n})$

Patterns

- The vast majority of friendly numbers found are abundant. The least index is $I(273) = 1.641$.
- No numbers tested had their smallest friend between 10^{20} and 10^{30}.
- All friendly numbers found were divisible by 2 or 3.
- No semiprime friendly numbers were found, other than 6.

Open Problems

1. Did we miss any?
2. Find a friendly number not divisible by 2 or 3.
3. Determine the status of any (non-6) semiprime pq where $q|(p+1)$.
4. Determine the asymptotic density of the friendly numbers.
5. Determine the lower bound of the abundancy index among friendly numbers.

Bibliography

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- R. Laatsch, *Measuring the Abundancy of Integers*, Mathematics Magazine Vol. 59, No. 2 (Apr. 1986), 84-92.
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