Abstract Algebra III

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Peer review: let me know if any concerns.

Today! HW 5 #1

→ this problem is hard

$P_1, P_2 \in \mathrm{Syl}_p(G)$

Fact about Sylow subgps

In G, for fixed p, any 2 Sylow p-subgps are conjugate, i.e. $\exists\, g \in G$ with $P_2 = gP_1g^{-1}$

Since any 2 Sylow p-subgroups are conjugate, they are isomorphic

The map $P_1 \to P_2$ is a gp isomorphism

say $P_2 = gP_1g^{-1}$ $\qquad h \mapsto ghg^{-1}$

If $P_1 \cong C_p \times C_p$ then $P_2 \cong C_p \times C_p$

a) $\exists g \in G$ with $p_2 = g p_1 g^{-1}$ $\Leftrightarrow$ $\exists h \in N_G(P)$ with $p_2 = h p_1 h^{-1}$

$p_1, p_2 \in P$

$\Leftarrow$ is clear because $h \in N_G(P) \leq G$

use $g = h$

<u>Not</u> if $p_2 = g p_1 g^{-1}$ $\not\Rightarrow$ $g \in N_G(P)$

Let's investigate how that can happen:

say $p_2 = g p_1 g^{-1}$ and also $p_2 = h p_1 h^{-1}$

$$g \neq h$$

then $h^{-1} \cdot g p_1 g^{-1} \cdot g = h^{-1} \cdot h p_1 h^{-1} \cdot g$ $\quad x = h^{-1} g \in C_G(p_1)$

$$\underbrace{(h^{-1}g)}_{x} p_1 = p_1 \underbrace{(h^{-1}g)}_{x}$$

if $gp_1g^{-1} = p_2$ then $hp_1h^{-1} = p_2$ also

iff $h^{-1}g \in C_G(p_1)$

$x = h^{-1}g \in C_G(p_1)$

i.e. every single h such that $hp_1h^{-1} = p_2$

is of the form $h = gx^{-1}$ for $x \in C_G(p_1)$

So now I'm looking for $x \in C_G(P_1)$

s.t. $gx^{-1} \in N_G(P)$

i.e. want $(gx^{-1}) P (gx^{-1})^{-1} = P$

$$g(x^{-1}Px)g^{-1} = P$$

$$x^{-1}Px = g^{-1}Pg$$

So now I want $x \in C_G(P_1)$ s.t.

$$x^{-1}Px = g^{-1}Pg = P_1$$

I know that P & P_1 are conjugate in G b/c they are both Sylow p-subgps of G.

I want to know that P and P_1 are conjugate in $C_G(P_1)$ $\rightarrow$ make sure that they are both Sylow p-subgps of $C_G(P_1)$

So now for $P_1 = g^{-1}Pg$, want to show that

$P_1, P \leq C_G(p_1)$ and they are Sylow p-subgps

given

Recall that P is abelian, so since $p_1 \in P$, $P \leq C_G(p_1)$

Let $q \in P_1 = g^{-1}Pg$, then $\exists\, p_3 \in P$ s.t. $q = g^{-1}p_3 g$

check: $q p_1 q^{-1} = (g^{-1}p_3 g) p_1 (g^{-1}p_3^{-1}g)$

want $\stackrel{?}{=} p_1$

$$gp_1g^{-1} = (g^{-1}p_3g)p_1(g^{-1}p_3^{-1}g)$$
$$= g^{-1}p_3(gp_1g^{-1})\ p_3^{-1}g$$
$$= g^{-1}(p_3\ p_2\ p_3^{-1})g$$
$$= g^{-1}\ p_2 g$$
$$= p_1$$

$$gp_1g^{-1} = p_2$$

$$p_3p_2p_3^{-1} = p_2$$
since P is abelian

Recap:

Let $g \in G$ be s.t. $gp_1g^{-1} = p_2$ for $p_1, p_2 \in P$ abelian

If $g \in N_G(P)$, done

Otherwise, let $P_1 = g^{-1}Pg$

Then $P, P_1 \leq C_G(p_1)$ (show this) and they are Sylow p-subgps of $C_G(p_1)$ (show this too!)

Let $x \in C_G(p_1)$ s.t. $x^{-1} P x = P_1$

Then $h = x^{-1} g \in N_G(P)$ (show this)

and $h p_1 h^{-1} = p_2$ (show this)

$\forall g \in \cancel{G}$ $C_G(P_1)$ gPg^{-1} is another Sylow p-subg of $\cancel{G}$ $C_G(P_1)$

$P \trianglelefteq G$ iff $gPg^{-1} = P$ $\forall g \in \cancel{G}$ $C_G(P_1)$

iff P is the only Sylow p-subgp

That's all for today!

We can do 1b) on Campuswire

→ a remix of 1a)