# Abstract Algebra III

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HW4 #1b)

a) $A \trianglelefteq G$, $B \trianglelefteq G$, $G/A$, $G/B$ both solvable then

$G/(A \cap B)$ is solvable

b) $G$ has a unique smallest subgp $G^{(\infty)}$

s.t. $G^{(\infty)} \trianglelefteq G$ and $G/G^{(\infty)}$ solvable

Suppose not,

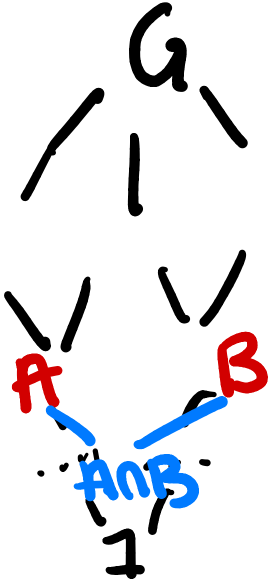
1) G is finite, G has finitely many subgps total

Not algebra: set $\{\frac{1}{n} : n \in \mathbb{Z}_{>0}\}$ has no smallest element

→ because of this, the way in which b) can fail is by having 2 subgps A, B both normal with solvable quotient, neither has a subgp with these props (normal w/ solvable quotient)

and $A \not\subseteq B$, $B \not\subseteq A$

i.e. "two local minima"




lattice of subgps of $G$

$\Rightarrow$ $A \cap B$ is normal with solvable quotient (by a)

$A \cap B < A$ $\quad$ $A \cap B < B$ $\quad$ contradiction

Another way to see this (not by contradiction)

Let $G^{(\infty)} = \bigcap_{\substack{A \trianglelefteq G \\ G/A \text{ solvable}}} A$

this is a finite intersection

by a) this is normal w/ solvable quotient and it contains all such gps.

HW 4 #3

Lemma: If $G = H \times K$ then $Z(G) = Z(H) \times Z(K)$.

proof: Let $(h,k) \in Z(H) \times Z(K)$

Let $(h_1, k_1) \in G$

$$(h,k)\cdot(h_1,k_1) = (hh_1, kk_1) = (h_1h, k_1k) = (h_1,k_1)(h,k)$$

so $(h,k) \in Z(G)$

Let $g \in Z(G)$ so $g=(h,k)$ $h \in H$, $k \in K$

Let $h_1 \in H$, $k_1 \in K$ $\rightsquigarrow$ $(h_1,k_1) \in G$

because $g \in Z(G)$, $(h,k)(h_1,k_1) = (h_1,k_1)(h,k)$

$$\begin{matrix} (h,k)(h_1,k_1) & = & (h_1,k_1)(h,k) \\ \| & & \| \\ (hh_1,kk_1) & & (h_1h,k_1k) \end{matrix}$$

this holds iff $hh_1 = h_1h$ and $kk_1 = k_1k$

$\Rightarrow h \in Z(H)$ $\qquad$ $\Rightarrow k \in Z(K)$

$\Rightarrow g=(h,k) \in Z(H) \times Z(K)$

~~#3~~ b) Understanding solvability

⇝ about understanding few facts about solvable gps

Section 3.4, 6.1

Key element for Christelle

Understanding the proof of the fact:

If $N \trianglelefteq G$ and $N$, $G/N$ are solvable then $G$ is solvable.

Recall that $G$ is solvable if we can

$$1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G_r = G$$

this alone can always be done $1 \trianglelefteq G$

and $\underbrace{G_i/G_{i-1} \text{ is abelian}}$ important extra assumption

abelian

$$\overline{G_i}/\overline{G_{i-1}} = \frac{(G_i/N)}{(G_{i-1}/N)} \cong G_i/G_{i-1} \quad \text{3rd}$$

$$1 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G$$

by 4th isom on $G_i$

$\overline{G_{i-1}} \trianglelefteq \overline{G_i}$ iff $G_{i-1} \trianglelefteq G_i$

abelian quotients

$\overline{G_i} = G_i/N$

abelian quotients

$$1 \trianglelefteq \overline{G_1} \trianglelefteq \overline{G_2} \trianglelefteq \dots \trianglelefteq G/N$$

# 4th isomorphism thm for gps

Check: $N \trianglelefteq G \Rightarrow N \trianglelefteq H$

bijection

$$\left\{\begin{array}{c}\text{subgps of } G \\ \text{containing } N\end{array}\right\} \longleftrightarrow \left\{\text{subgps of } G/N\right\}$$

$$H \mapsto \bar{H} = H/N$$

and $\bar{H} \trianglelefteq G/N$ iff $H \trianglelefteq G$

Also! by 3rd isom $\quad G/H \cong (G/N)/(H/N)$

# 4th isomorphism thm for gps

Check: $N \trianglelefteq G \Rightarrow N \trianglelefteq H$

bijection

$$\left\{ \begin{array}{c} \text{subgps of } G \\ \text{containing } N \end{array} \right\} \longleftrightarrow \left\{ \text{subgps of } G/N \right\}$$

$G/N = \overline{G}$

$H \mapsto \overline{H} = H/N$

and $\overline{H} \trianglelefteq G/N$ ($= \overline{G}$) iff $H \trianglelefteq G$

Also! by 3rd isom $G/H \cong (G/N)/(H/N)$

HW4 #1(a)

$G/A$ solvable

$G/B$ solvable

$1 \trianglelefteq \bar{A}_1 \trianglelefteq \bar{A}_2 \trianglelefteq \dots \trianglelefteq G/A$

$A \cap B \trianglelefteq$ ?? $\dots \trianglelefteq A \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq \dots \trianglelefteq G$

$B \trianglelefteq B_1 \trianglelefteq \dots \underbrace{\trianglelefteq G}_{\text{too long}}$

use 2nd isom theorem

a subgp of a solvable gp is solvable

# That’s all for today!