Abstract Algebra III

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

HW1 on Gradescope last Friday

HW2 should be with peer reviewer

Now HW3 #1-4 group actions

#5 automorphisms

#6 ~ p-gps

HW 2 #3c)

G - finite, $\#G = n$

$\pi : G \to S_n$ left reg rep

a) n even $\Rightarrow$ G contains an element 2

Cauchy's Thm: If p is a prime and $p \mid \#G$ then G contains an element of order p.

b) n is even $x \in G$ x of order 2

$\pi(x) \in S_n$ is a product of $\frac{n}{2}$ transpositions

c) If $n = 2m$, m odd then G has a normal subgp of index 2.

$$G \xrightarrow{\pi} S_n \xrightarrow{sgn} \{\pm 1\} \cong C_2$$

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Every $\sigma \in S_n$ can be written <u>uniquely</u> as a product of <u>disjoint</u> cycles

Overall I have a gp hom $\varphi : G \to C_2$

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Grant for a minute that this gp hom is nontrivial; i.e., $\exists\, g \in G$ with $\varphi(g) \neq id$

(will prove this after)

If so, $Im\,\varphi = C_2$ and by 1st isom theorem

$$G/\ker\varphi \cong Im\,\varphi \cong C_2$$

$G/\ker\varphi \cong C_2$ $\qquad \boxed{\frac{\#G}{\#\ker\varphi} = 2}$

kernels always normal

i.e. $\ker\varphi$ is a normal subgp of index 2

since $\#G = \#\ker\varphi \, \underbrace{[G:\ker\varphi]}_{=2}$

Claim: $\operatorname{sgn}(\pi(x)) = -1$ $\quad x$ is from b)

$\pi(x)$ is a product of $\frac{n}{2} = m$ odd

transpositions (2-cycle) so

$$\operatorname{sgn}(\pi(x)) = \Big(\underbrace{(-1)^{2-1}}_{\text{sgn of 1 transposition}}\Big)^m = (-1)^m = -1$$

HW 2 #1a) Let $\underbrace{g \in G_w \text{ then } \boxed{hgh^{-1}} \in \underline{G_{hw}}}$

$\Updownarrow$

$hG_wh^{-1} \subseteq G_{hw}$

$$\begin{aligned}
(hgh^{-1})\cdot(hw) &= [(hg(h^{-1})\cdot h)]\cdot w \\
&= (hg)\cdot w \\
&= h\cdot(g\cdot w) = h\cdot w \\
&= hw \quad \leftarrow \text{want}
\end{aligned}$$

✓

HW #1c) S_3 acts on S_3 by mult on the left

then $N = \{id\} = G_w$

$S_3 \curvearrowright \{1,2,3\}$ $N = \{id\}$

$G_1 = \{id, (23)\}$

$G_2 = \{id, (13)\}$

$G_3 = \{id, (12)\}$

HW2 #3b) $G \curvearrowright G$ by left mult.

$x \in$ order 2

$\pi: G \to S_n$
$x \mapsto ???$

Under any φ a gp hom the order of $\varphi(g)$ divides the order of g

proof: Let $g^k = 1$, g of order k, then

$1 = \varphi(1) = \varphi(g^k) = \varphi(g)^k \Rightarrow \varphi(g)^k = 1$

$\Rightarrow$ order $\varphi(g)$ divides k

$\pi : G \to S_n$
$x \mapsto \pi(x)$

since x is of order 2
$\pi(x)$ is of order 1 or 2
(Note that only id has order 1)

Is there $g \in G$ with $x \cdot g = g$?

or: Is there any $g \in G$ fixed by x?

No if $x \cdot g = g$ then $x \cdot g \cdot g^{-1} = g \cdot g^{-1}$
$x = id$

but x is of order 2 so $x \neq id$.

So $\forall g \in G \quad x \cdot g \neq g$

$$G = \left\{ \begin{array}{l} \cancel{id}\ 1 \\ \cancel{g}\ 2 \\ \cancel{h}\ 3 \\ \vdots\ 4 \\ \quad 5 \end{array} \right\}$$

$$\pi : G \to S_n$$
$$x \mapsto \pi(x)$$

$\pi(x)$ doesn't fix any number

decomposition of $\pi(x)$

Every number $1, 2, 3, \ldots, n$ appears in the cycle

$\sigma = (12)(34) \in S_5$

σ fixes 5

Aside

$\sigma = (12)(345)$

$\pi(x)$ contains $\{1, \ldots, n\}$ in its cycle decomp.

In fact: When σ is written (uniquely) as a product of disjoint cycles, its order is the lcm of the lengths of the cycles appearing

So $\pi(x)$ is of order 2 $(\pi(x) \neq id)$

So it must be a product of disjoint <u>transpositions</u>

but every number must appear

$\Rightarrow \frac{n}{2}$ transpositions

$\pi(x) = (13)(45)(26) \in S_6$

2 elements in S_n are conjugate to each other iff they have the same "cycle type"

In S_4

$(12)(34)$ and $(13)(24)$ conjugates

$(12)(34)$ and (123) not conjugates

$(12)(34) \qquad (13)(24)$

conjugates by $\tau = (12)$

$$\{(123),\ (12)(34)\} \to \{(132),\ (14)(23)\}$$

That’s all for today!