Math 395 - Fall 2020
Quiz 9

Please solve **ONE** of the three problems below:

1. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and let $\alpha = \sqrt{2} - \sqrt{3}$.
   (a) Show that $[L(\sqrt{\alpha}) : L] = 2$ and $[L(\sqrt{\alpha}) : \mathbb{Q}] = 8$.
   (b) Find the minimal polynomial of $\sqrt{\alpha}$ over $\mathbb{Q}$.
   (c) Show that $L(\sqrt{\alpha})$ is not Galois over $\mathbb{Q}$.
2. Let α be the real, positive fourth root of 5, and let $i = \sqrt{-1} \in \mathbb{C}$. Let $K = \mathbb{Q}(\alpha, i)$.
   (a) Prove that $K/\mathbb{Q}$ is a Galois extension with Galois group dihedral of order 8.
   (b) Find the largest abelian extension of $\mathbb{Q}$ in K (i.e., the unique largest subfield of K that is Galois over $\mathbb{Q}$ with abelian Galois group) – justify your answer.
   (c) Show that $\alpha + i$ is a primitive element for $K/\mathbb{Q}$.
3. Let F/E be a Galois extension of degree 4, where E and F are fields of characteristic different from 2. Show that $\mathrm{Gal}(F/E) \cong C_2 \times C_2$ if and only if there exist $x, y \in E$ such that $F = E(\sqrt{x}, \sqrt{y})$ and none of x, y or xy are squares in E.