Math 395 - Fall 2020
Quiz 3

Please solve **ONE** of the three problems below:

1. Let G be a finite group acting transitively on the left on a nonempty set Ω. Let $N \trianglelefteq G$, and let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the orbits of N acting on Ω. For any $g \in G$, let

$$g\mathcal{O}_i = \{g\alpha : \alpha \in \mathcal{O}_i\}.$$

   (a) Prove that $g\mathcal{O}_i$ is an orbit of N for any $i \in \{1, 2, \ldots, r\}$, i.e., $g\mathcal{O}_i = \mathcal{O}_j$ for some j.

   (b) With G acting as in part (a), explain why G permutes $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ transitively.

   (c) Deduce from (b) that $r = [G : NG_\alpha]$, where G_α is the subgroup of G stabilizing the point $\alpha \in \mathcal{O}_1$.

2. Let N be a normal subgroup of the group G, and for each $g \in G$, let ϕ_g denote conjugation by g acting on N, i.e,

$$\phi_g(x) = gxg^{-1} \quad \text{for all } x \in N.$$

   (a) Prove that ϕ_g is an automorphism of N for each $g \in G$.

   (b) Prove that the map $\Phi\colon g \mapsto \phi_g$ is a homomorphism from G into $\mathrm{Aut}(N)$.

   (c) Prove that $\ker \Phi = C_G(N)$ and deduce that $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$.

3. In this problem, G is a finite group.

   (a) Show that if $G/Z(G)$ is cyclic, where $Z(G)$ is the center of G, then G is abelian.

   (b) Let p be a prime and P be a p-group. Show that $Z(P)$ is nontrivial.

   (c) Show that if P has order p^2 then P is abelian.

   (d) Show that every p-group is solvable.