
Abstract Algebra III

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HWS #4c)

G finite such that every Sylow p -subgroup is abelian

Show that G is not simple

$\exists H < G$ such that $H \triangleleft G$
proper

If G is abelian, then every subgroup is normal
so if G has a proper subgroup, then G is
not simple

$G = C_p$ p prime abelian + simple
bc no proper subgroup

Suppose that G is not abelian

$1 < H < G$, H abelian

~~G is not simple~~

If $N \trianglelefteq H$, $H \leq G \Rightarrow N$ may or may not be normal in G

If $N \triangleleft H$ and $H \triangleleft G$ ~~$N \triangleleft G$~~

But if $N \text{ char } H$ $H \triangleleft G$ then $N \triangleleft G$

Claim: If $N \text{ char } H$ and $H \triangleleft G$ then $N \triangleleft G$

proof: Recall that $N \text{ char } H$ means that

$\forall \varphi \in \text{Aut}(H), \varphi(N) = N$ setwise

Since $H \triangleleft G$ every $g \in G$ gives rise to an element of $\text{Aut}(H)$ by conjugation

$$\varphi_g(h) = ghg^{-1} \quad \varphi_g \in \text{Aut}(H)$$

Therefore for $g \in G$ $gNg^{-1} = \varphi_g(N) \stackrel{\uparrow}{=} N$
N char H

so $N \trianglelefteq G$.

Idea deep down: IF $N \trianglelefteq H$ but not char
then $\varphi_g \in \text{Aut}(H)$ might be an aut of H
that doesn't fix N .

$N \trianglelefteq H$ means that $\varphi_h(N) = N$ but maybe not all φ_g

Characteristic subgps

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$$

$$\forall y \quad xy = yx$$

$\Downarrow \varphi(x) \in Z(G)$
bc $\varphi(y)$ runs thru
all elts of G

• $Z(G)$ char G

• If $P \in \text{Syl}_p(G)$ and $P \trianglelefteq G$ then P char G
 $n_p = 1$

• $H = \{g \in G : \text{ord}(g) = p \text{ prime}\} \cup \{1\}$, H char G

• Commutator subgp G' char G .

$$G' = \langle \{xyx^{-1}y^{-1} : x, y \in G\} \rangle$$

$$\varphi \in \text{Aut}(G)$$

$$\varphi(xyx^{-1}y^{-1}) = \underbrace{\varphi(x)}_g \underbrace{\varphi(y)}_h \underbrace{\varphi(x^{-1})}_{g^{-1}} \underbrace{\varphi(y^{-1})}_{h^{-1}}$$

HW 6 #1 G simple $\#G = 3^2 \cdot 7^2 \cdot 11$

a) $n_3 \equiv 1 \pmod{3}$

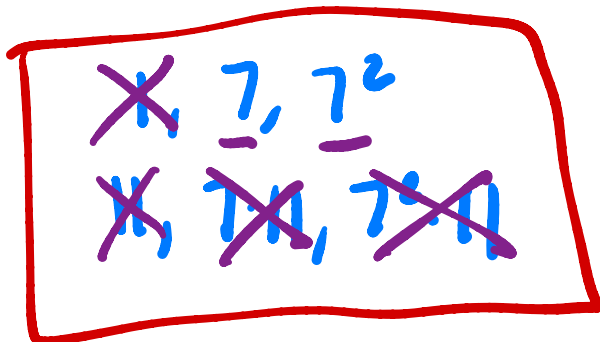
$n_3 \mid 7^2 \cdot 11$

$n_3 = 7, 7^2$

$= [G : N_G(P_3)]$

$= \frac{\#G}{\#N_G(P_3)}$

divisors of $7^2 \cdot 11$ are of
the form $7^i \cdot 11^j$ $i=0,1,2$
 $j=0,1$



$P_3 \in \text{Syl}_3(G)$

$7 \cdot 11 \pmod{3}$

$\equiv 1 \cdot 2 \pmod{3}$

$\equiv 2 \pmod{3}$

$7^2 \cdot 11 \equiv 1 \cdot 1 \cdot 2 \pmod{3}$

$\equiv 2 \pmod{3}$

if $n_3 = 7$ $\#N_G(P_3) = 3^2 \cdot 7 \cdot 11$; if $n_3 = 7^2$, $\#N_G(P_3) = 3^2 \cdot 11$

$$n_7 \equiv 1 \pmod{7}$$

$$n_7 \mid 3^2 \cdot 11$$

divisors

~~1, 3, 3²~~

~~11, 3²·11~~, 3²·11

$$n_7 = 3^2 \cdot 11$$

$$3 \cdot 11 \equiv 3 \cdot 4 \equiv 12 \equiv 5 \pmod{7}$$

$$3^2 \cdot 11 \equiv 2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}$$

$$\#N_G(P_7) = 7^2$$

$$(\#P_7 = 7^2)$$

$$n_{11} \equiv 1 \pmod{11}$$

$$n_{11} \mid 3^2 \cdot 7^2$$

$$n_{11} = 3^2 \cdot 7^2$$

$$\#N_G(P_{11}) = 11$$

~~$$1, 3, 9$$~~

~~$$7, 3 \cdot 7, 3^2 \cdot 7$$~~

~~$$7^2, 3 \cdot 7^2, 3^2 \cdot 7^2$$~~

$$\equiv -2 \cdot 5 \equiv -10 \equiv 1 \pmod{11}$$

b) Show that $\exists P \neq Q \in \text{Syl}_7(G)$ with

$$\# P \cap Q = 7$$

Recall that $\# P = \# Q = 49$

so either

$$\# P \cap Q = 1$$

$$\# P \cap Q = 7$$

~~$$\# P \cap Q = 49 \quad P = Q$$~~

By contradiction assume that if $P \neq Q$ then
 $P \cap Q = 1$

If so let's count the number of elements of G that are not the identity and have order a power of 7

We have 99 Sylow 7-subgroups

Each contains 48 elements of order a power of 7

$$\Rightarrow \text{get } 3^2 \cdot 11 \cdot (7^2 - 1) = 3^2 \cdot 7^2 \cdot 11 - 3^2 \cdot 11$$

elements of order a power of 7

This leaves only $3^2 \cdot 11$ other elements in G
in total, "99

$$P_{11} \cong C_{11}$$

$$P, Q \in \text{Syl}_{11}(G)$$

But we have $3^2 \cdot 7^2$ Sylow 11-subgroups $P \cap Q = 1$
each have 10 elements of order 11 OR $P = Q$

so $3^2 \cdot 7^2 \cdot 10$ elements of order 11 in total
" $21 \cdot 210 > 99$

contradiction.

$$c) H = P \cap Q \quad \#H = 7 \quad P, Q \in \text{Syl}_7(G)$$

- P, Q are abelian because they have size p^2 for p a prime
 (you should be able to prove this but here you can assume it)

$$\begin{array}{ccccccc}
 H & \leq & P, Q & \leq & C_G(H) & \leq & N_G(H) & \leq & G \\
 \uparrow & & & & & & & & \\
 & & 7^2 & & \underline{3^? \cdot 7^2 \cdot 11^?} & & \underline{3^? \cdot 7^2 \cdot 11^?} & & 3^2 \cdot 7^2 \cdot 11
 \end{array}$$

this index cannot be 1 or 3

this index cannot be 1 or 3

$$\begin{array}{ccccccc}
 H & \leq & P, Q & \leq & C_G(H) & \leq & N_G(H) & \leq & G \\
 112 & & 7^2 & & 3^? \cdot 7^2 \cdot 11^? & & 3^0 \cdot 7^2 \cdot 11^1 & & 3^2 \cdot 7^2 \cdot 11 \\
 G & & & & & & & &
 \end{array}$$

By contradiction $11 \mid \#N_G(H)$

Let $g \in N_G(H)$ with g of order 11 (exists by Cauchy's Thm)

$$gHg^{-1} = H \quad \varphi_g \in \text{Aut}(H) \cong C_6$$

$$N_G(H) \xrightarrow{\text{gp hom}} \text{Aut}(H) \cong C_6$$

g
order 11

only elements of
order 1, 2, 3, 6

$\forall g$ has order dividing
11

$\Rightarrow \forall g$ has order 1

i.e. $g \in C_G(H)$

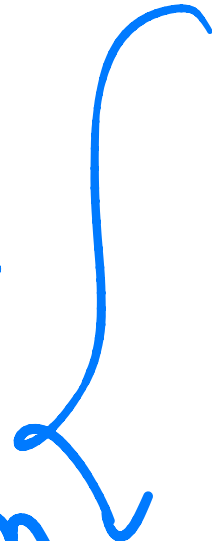
Sylow 7-subgps of $C_G(H)$

$$\#C_G(H) = 7^2 \cdot 11$$

$$n_7(C_G(H)) = 1$$

but $P, Q < C_G(H)$

$P \neq Q$ contradiction
2 distinct



FOR a) See if $\# N_G(H) = 3 \cdot 7^2$ OR $3^2 \cdot 7^2$
OR 7^2 work

That's all for today!

OH 4pm-5pm

(4-4:15 taken)

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