Abstract Algebra III

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HW5 #4c)

G finite such that every Sylow p-subgp is abelian

Show that G is not simple

$\exists \underset{\text{proper}}{\underline{H < G}}$ such that $H \trianglelefteq G$

If G is abelian, then every subgp is normal
so if G has a proper subgp, then G is not simple

$G = C_p$ p prime   abelian + simple
bc no proper subgp

Suppose that G is not abelian

$1 < H < G$, H abelian

$\not\Rightarrow$ G is not simple

If $N \trianglelefteq H$, $H \leq G$ $\Rightarrow$ N may or may not be normal in G

If $N \ntrianglelefteq H$ and $H \ntrianglelefteq G$ $\not\Rightarrow$ $N \ntrianglelefteq G$

<u>But</u> if N char H $H \triangleleft G$ then $N \triangleleft G$

Claim: If N char H and $H \trianglelefteq G$ then $N \trianglelefteq G$

proof: Recall that N char H means that

$\forall\ \varphi \in \mathrm{Aut}(H)$, $\varphi(N) = N$ setwise

Since $H \trianglelefteq G$ every $g \in G$ gives rise to an element of $\mathrm{Aut}(H)$ by conjugation

$$\varphi_g(h) = ghg^{-1} \qquad \varphi_g \in \mathrm{Aut}(H)$$

Therefore for $g \in G$ $\quad gNg^{-1} = \varphi_g(N) \underset{\substack{\uparrow \\ N \text{ char } H}}{=} N$

so $N \trianglelefteq G$.

Idea deep down: If $N \trianglelefteq H$ but not char then $\varphi_g \in \mathrm{Aut}(H)$ might be an aut of H that doesn't fix N.

$N \trianglelefteq H$ means that $\varphi_h(N) = N$ but maybe not all φ_g

Characteristic subgps

- $Z(G)$ char G

  $\forall y \quad xy = yx$

  $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$

  $\Downarrow \quad \varphi(x) \in Z(G)$

  bc $\varphi(y)$ runs thru all elts of G

- If $P \in Syl_p(G)$ and $P \trianglelefteq G$ then P char G

  $n_p = 1$

- $H = \{ g \in G : ord(g) = p \text{ prime} \} \cup \{1\}$, H char G

- Commutator subgp G' char G.

$G' = \langle \{ xyx^{-1}y^{-1} : x, y \in G \} \rangle$

$\varphi \in Aut(G)$

$$\varphi(xyx^{-1}y^{-1}) = \underbrace{\varphi(x)}_{g}\underbrace{\varphi(y)}_{h}\underbrace{\varphi(x)^{-1}}_{g^{-1}}\underbrace{\varphi(y)^{-1}}_{h^{-1}}$$

HW 6 #1 $\quad G$ simple $\quad \#G = 3^2 \cdot 7^2 \cdot 11$

a) $n_3 \equiv 1 \bmod 3$

$n_3 \mid 7^2 \cdot 11$

divisors of $7^2 \cdot 11$ are of the form $7^i \cdot 11^j$ $\quad i = 0,1,2$, $j = 0,1$

$$\boxed{\cancel{1},\ 7,\ 7^2 \\ \cancel{11},\ \cancel{7 \cdot 11},\ \cancel{7^2 \cdot 11}}$$

$7 \cdot 11 \bmod 3$
$\equiv 1 \cdot 2 \bmod 3$
$\equiv 2 \bmod 3$

$7^2 \cdot 11 \equiv 1 \cdot 1 \cdot 2 \bmod 3$
$\equiv 2 \bmod 3$

$$n_3 = 7, 7^2$$
$$= [G : N_G(P_3)]$$
$$= \frac{\#G}{\#N_G(P_3)}$$

$P_3 \in Syl_3(G)$

if $n_3 = 7$ $\quad \#N_G(P_3) = 3^2 \cdot 7 \cdot 11$; if $n_3 = 7^2$, $\#N_G(P_3) = 3^2 \cdot 11$

$n_7 \equiv 1 \mod 7$

$n_7 \mid 3^2 \cdot 11$

divisors $\cancel{1}$, $\cancel{3}$, $\cancel{3^2}$,

$\cancel{11}$, $\cancel{3 \cdot 11}$, $3^2 \cdot 11$

$3 \cdot 11 \equiv 3 \cdot 4 \equiv 12 \equiv 5 \mod 7$

$n_7 = 3^2 \cdot 11$

$3^2 \cdot 11 \equiv 2 \cdot 4 \equiv 8 \equiv 1 \mod 7$

$\# N_G(P_7) = 7^2$

$(\# P_7 = 7^2)$

$n_{11} \equiv 1 \mod 11$

$n_{11} \mid 3^2 \cdot 7^2$

$n_{11} = 3^2 \cdot 7^2$

$\# N_G(P_{11}) = 11$

$\cancel{1}$, $\cancel{3}$, $\cancel{9}$

$\cancel{7}$, $\cancel{3 \cdot 7}$, $\cancel{3^2 \cdot 7}$

$\cancel{7^2}$, $\cancel{3 \cdot 7^2}$, $3^2 \cdot 7^2$

$\equiv$

$-2 \cdot 5 \equiv -10 \equiv 1 \mod 11$

b) Show that $\exists P \neq Q \in Syl_7(G)$ with

$\# P \cap Q = 7$

Recall that $\# P = \# Q = 49$

so either

$\# P \cap Q = 1$

$\# P \cap Q = 7$

$\cancel{\# P \cap Q = 49 \quad P = Q}$

By contradiction assume that if $P \neq Q$ then

$P \cap Q = 1$

If so let's count the number of elements of G that are not the identity and have order a power of 7

We have 99 Sylow 7-subgps

Each contains 48 elements of order a power of 7

$\Rightarrow$ get $3^2 \cdot 11 \cdot (7^2 - 1) = 3^2 \cdot 7^2 \cdot 11 - 3^2 \cdot 11$ elements of order a power of 7

This leaves only $3^2 \cdot 11$ other elements in G
in total. $\;\; = 99$

$P_{11} \cong C_{11}$

$P, Q \in \mathrm{Syl}_{11}(G)$

<u>But</u> we have $3^2 \cdot 7^2$ Sylow 11-subgps
each have 10 elements of order 11

$P \cap Q = 1$
or $P = Q$

So $3^2 \cdot 7^2 \cdot 10$ elements of order 11 in total
$= 21 \cdot 210 > 99$

contradiction.

c) $H = P \cap Q$ $\quad \#H = 7 \qquad P, Q \in Syl_7(G)$

• P, Q are abelian because they have size p^2 for p a prime (you should be able to prove this but here you can assume it)

this index cannot 1 or 3

$$\begin{array}{ccccccccc} H & \lneq & P, Q & \leq & C_G(H) & \leq & N_G(H) & \leq & G \\ & & 7^2 & & 3^? \underline{7^2} 11^? & & 3^? \underline{7^2} 11^? & & 3^2 \cdot 7^2 \cdot 11 \end{array}$$

this index cannot 1 or 3

$$\underset{11^2}{H} \not\leq \underset{7^2}{P, Q} \leq \underset{\underline{3^{?}7^2}\,11^{?}}{C_G(H)} \leq \underset{\underline{3^{0}7^2}\,11^{1}}{N_G(H)} \leq \underset{3^2\cdot 7^2\cdot 11}{G}$$

C_7

By contradiction $11 \mid \# N_G(H)$

Let $g \in N_G(H)$ with g of order 11 (exists by Cauchy's Thm)

$gHg^{-1} = H \qquad \varphi_g \in \operatorname{Aut}(H) \cong C_6$

$$N_G(H) \xrightarrow{\text{gp hom}} \mathrm{Aut}(H) \cong C_6$$

g

order 11

only elements of order 1, 2, 3, 6

φ_g has order dividing 11

$\Rightarrow \varphi_g$ has order 1

i.e., $g \in C_G(H)$

2 distinct Sylow 7-subgps of $C_G(H)$

$\Leftarrow$

$\# C_G(H) = 7^2 \cdot 11$

$n_7(C_G(H)) = 1$

but $P, Q < C_G(H)$

$P \neq Q$ contradiction

FOR d) See if $\# N_G(H) = 3 \cdot 7^2$ OR $3^2 \cdot 7^2$

OR 7^2 WORK

That's all for today!

OH 4pm-5pm

(4-4:15 taken)

Campuswire