
Abstract Algebra III

— This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat. —

2 chapters in D&F on field theory

linear algebra / vector spaces Chap 11 11.1, 11.2

Chapter 13: Basic stuff / Background /

Foundational ideas

Chapter 14: Galois theory

This is the study of certain special, very
nice field extensions,

"Galois extension"

13.1 Basic theory of fld extensions

What if we have a fld F and a polynomial $p(x) \in F[x]$ and we want to make a bigger field K in which p has a root

Let α be a root of $p(x)$

$$F(\alpha) \cong F[x]/(p(x))$$

adjoin α to F

this only works if p is irreducible!

If p is not irreducible then $F[x]/(p(x))$
is not a field.

Analogy: $\mathbb{Z}/p\mathbb{Z}$ is a field if p is prime
 $\mathbb{Z}/n\mathbb{Z}$ is not a field if n is composite

So if p not irred, factor it $p(x) = \prod_{i=1}^n p_i(x)$

p_i irred then do

$$F(\alpha) = F[x]/(p_i(x))$$

13.2 Algebraic extensions

Definition: Let K/F be a fld extension

Then $\alpha \in K$ is algebraic over F if α

is a root of a polynomial $p(x) \in F[x]$

Examples: $\sqrt[3]{2}$ is alg / \mathbb{Q} $x^3 - 2 \in \mathbb{Q}[x]$

π is not alg / \mathbb{Q}

not algebraic, transcendental

Definition: K/F is an algebraic extension if

$\forall \alpha \in K$, α is alg / F

← over

Prop 9: If $\alpha \in K$ is alg / F , then \exists a unique
monic irreducible polynomial $m_{\alpha, F} \in F[x]$
with $m_{\alpha, F}(\alpha) = 0$

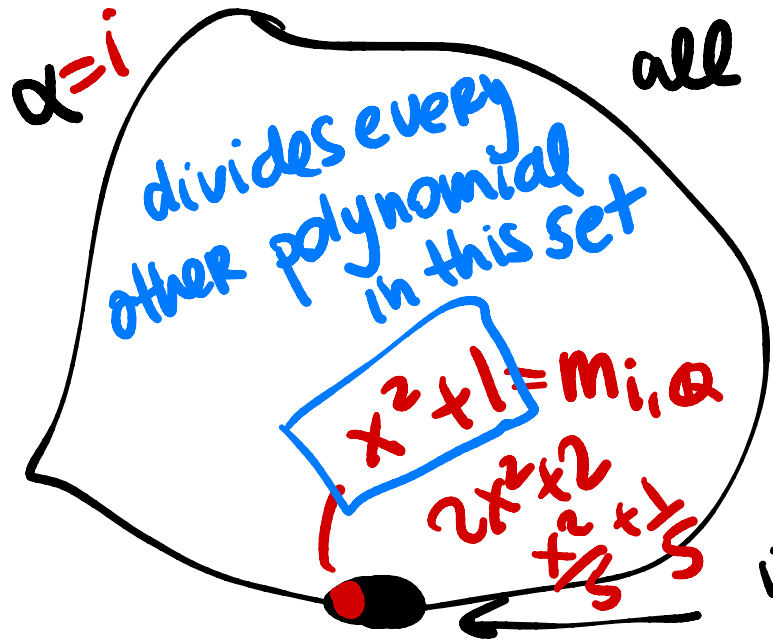
$m_{\alpha, F}$ is called the minimal polynomial of α
over F

If $f(x) \in F[x]$ and $f(a) = 0$ then

$$m_{a,F} \mid f \quad \text{in } F[x]$$

$$\deg f \geq \deg m_{a,F}$$

all polys
with $p(a) = 0$
 $p(x) \in F[x]$



irreducible
poly

← exciting that it's
irreducible

In math/algebra a minimal polynomial
is not always irreducible
(but it's always of least degree)

In field theory, the min poly of an
algebraic element is always irreducible

Quick Corollary (Proposition 11)

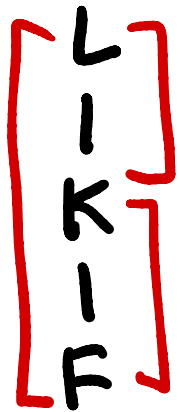
$$F(\alpha) \cong F[x] / (m_{\alpha, F(x)})$$

$$[F(\alpha) : F] = \deg m_{\alpha, F}$$

Prop 12 α is alg / F iff $[F(\alpha) : F] < \infty$

$$\Rightarrow [\mathbb{Q}(\pi) : \mathbb{Q}] = \infty$$

Theorem 14
 $F \subseteq K \subseteq L$



$$[L:F] = [L:K][K:F]$$

13.3 Straightedge + compass constructions
historical + fun

13.4 Splitting fields + alg closures

F a field

$f(x) \in F[x]$ be irreducible

Definition Let $p(x) \in F[x]$ be a polynomial, not necessarily irreducible. K/F is the splitting field of p if

- ① p factors completely in $K[x]$
into linear factors
- ② p does not factor completely in any $F \subseteq E \subsetneq K$

K is the smallest field containing F in which p "splits completely"

↳ factors into linear factors

Example: $x^3 - 1 \in \mathbb{Q}[x]$ is not irreducible but does not split completely

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

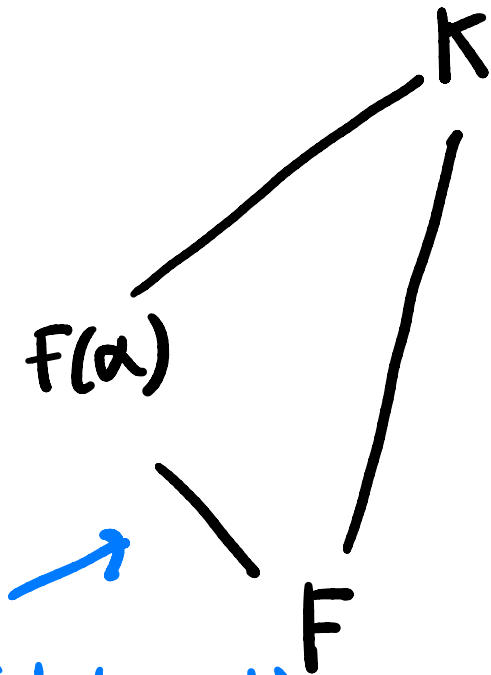
↳ not linear

F field

$f(x) \in F[x]$ irreducible

$\deg f = n$

$$F[x]/(f(x)) = F(\alpha)$$



this has
one root of f (at least)
this has degree n

2 important
extensions associated
to f

splitting field of f
this has all roots of
 f
this has degree dividing
 $n!$

in general must
compute degree

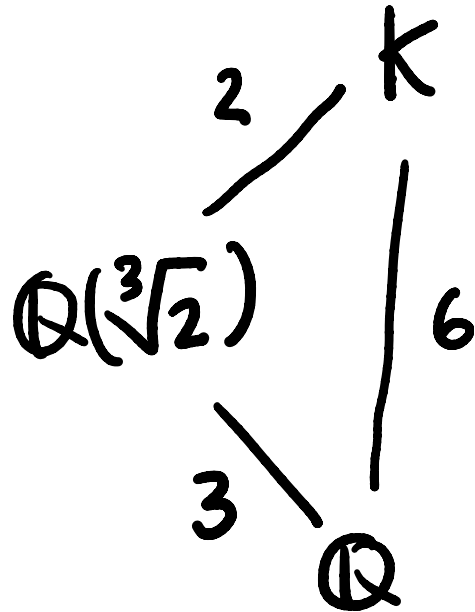
Example $f(x) = x^3 - 2 \in \mathbb{Q}[x]$ irreducible, monic

$$= m_{\sqrt[3]{2}, \mathbb{Q}}$$

In $\mathbb{Q}(\sqrt[3]{2})[x]$

$$x^3 - 2 = (x - \sqrt[3]{2}) \cdot$$

$$(x^2 + \sqrt[3]{2}x + \sqrt[3]{4})$$



splitting field

$$K = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$$

Theorem 25 + Corollary 28

For any $f \in F[x]$, the splitting field K of f over F exists and is unique up to isomorphism.

Definition: let F be a field, then we denote by \bar{F} the extension of F such that \bar{F} is algebraic over F and every polynomial $f(x) \in F[x]$ splits completely in $\bar{F}[x]$

(Prop 31 \bar{F} exists and is unique up to isom)

$\bar{F} = F +$ all algebraic elements over F

Definition: We say that K is algebraically closed
if every $p(x) \in K[x]$ splits completely in
 K already $(\Leftrightarrow K = \bar{K})$

Proposition \bar{K} is alg closed.

\mathbb{C} algebraically closed

$$\overline{\mathbb{R}} = \mathbb{C} \quad [\overline{\mathbb{R}} : \mathbb{R}] = 2 \quad \mathbb{C} = \mathbb{R}(i)$$

$$\overline{\mathbb{Q}} \neq \mathbb{C} \quad [\overline{\mathbb{Q}} : \mathbb{Q}] = \infty$$

$\searrow \pi, e$

$$i \in \overline{\mathbb{Q}} \not\subseteq \mathbb{R}$$

just \mathbb{Q} + algebraic elements over \mathbb{Q}
no transcendental elements

That's all for today!