

Math 395 - Fall 2020
Homework 5

This homework is due on Friday, October 2 to your peer reviewer, and on Friday, October 9 on Gradescope.

1. Let G be a finite group, let p be a prime and let $P \in \text{Syl}_p(G)$. Assume that P is abelian.
 - (a) Prove that two elements of P are conjugate in G if and only if they are conjugate in $N_G(P)$.
 - (b) Prove that $P \cap gPg^{-1} = 1$ for every $g \in G - N_G(P)$ if and only if $P \trianglelefteq C_G(x)$ for every nonidentity element $x \in P$.

2. Let G be a group of odd order and let σ be an automorphism of G of order 2.
 - (a) Prove that for every prime p dividing the order of G there is some Sylow p -subgroup P of G such that $\sigma(P) = P$ (i.e., σ stabilizes the subgroup P – note that σ need not fix P elementwise).
 - (b) Suppose that G is a cyclic group. Prove that $G = A \times B$ where

$$A = C_G(\sigma) = \{g \in G : \sigma(g) = g\} \quad \text{and} \quad B = \{x \in G : \sigma(x) = x^{-1}\}.$$

(Remark: This decomposition is true more generally when G is abelian.)

3. Let G be a finite group with the property that the centralizer of every nonidentity element is an *abelian* subgroup of G . (Such a group is called a *CA*-group.)
 - (a) Prove that every Sylow p -subgroup of G is abelian, for every prime p .
 - (b) Prove that if P and Q are distinct Sylow subgroups of G , then $P \cap Q = 1$.
4. Let p and q be distinct primes and let G be a group of order p^3q .
 - (a) Show that if $p > q$ then a Sylow p -subgroup of G is normal in G .
 - (b) Assume G has more than one Sylow p -subgroup. Show that if the intersection of any pair of distinct Sylow p -subgroup is the identity, then G has a normal Sylow q -subgroup.
 - (c) Assume the Sylow p -subgroups of G are abelian. Show that G is not a simple group. (Do not quote Burnside's $p^a q^b$ -theorem.)

5. Let G be a group of order 2457 (note that $2457 = 3^3 \cdot 7 \cdot 13$).
 - (a) Compute the number n_p of Sylow p -subgroups permitted by Sylow's Theorem for $p = 7$ and $p = 13$ (only).

- (b) Let P_{13} be a Sylow 13-subgroup of G . Prove that if P_{13} is not normal in G , then $N_G(P_{13})$ has a normal Sylow 7-subgroup.
- (c) Deduce from (b) and (a) that G has a normal Sylow p -subgroup for either $p = 7$ or $p = 13$.
6. Let G be a group of order 6545 (note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$).
- (a) Compute the number n_p of Sylow p -subgroups permitted by Sylow's Theorem for $p = 5$ and $p = 17$ (only).
- (b) Let P_5 be a Sylow 5-subgroup of G . Prove that if P_5 is not normal in G , then $N_G(P_5)$ has a normal Sylow 17-subgroup. (Keep in mind that $P_5 \trianglelefteq N_G(P_5)$.)
- (c) Deduce from (b) and (a) that G has a normal Sylow p -subgroup for either $p = 5$ or $p = 17$.
- (d) Deduce from (c) that $Z(G) \neq 1$.