Math 395 - Fall 2020
Homework 4

This homework is due on Friday, September 25 to your peer reviewer, and on Friday, October 2 on Gradescope.

1. Let G be a finite group.

   (a) Suppose that A and B are normal subgroups of G and both G/A and G/B are solvable. Prove that $G/(A \cap B)$ is solvable.

   (b) Deduce from (a) that G has a subgroup that is the unique smallest subgroup with the properties of being normal with solvable quotient – this subgroup is denoted $G^{(\infty)}$. (In other words, show that there is a subgroup $G^{(\infty)} \trianglelefteq G$ with $G/G^{(\infty)}$ solvable, and if G/N is any solvable quotient of G, then $G^{(\infty)} \leq N$.)

   (c) If G has a subgroup S isomorphic to A_5 (S is not necessarily normal), show that $S \leq G^{(\infty)}$.

   Note that if G is solvable, then $G^{(\infty)} = 1$, and if G is perfect, then $G^{(\infty)} = G$.

2. Let G be a finite group and p be a prime. Assume that G has a normal subgroup of order p, which we will call H.

   (a) Prove that if p is the smallest prime dividing the order of G, then H is contained in the center of G.

   (b) Prove that if G/H is a non-abelian simple group, then H is contained in the center of G.

3. Let $G = D_4 \times S_3$.

   (a) Find the center of G.

   (b) Is G solvable? Explain.

4. Let G be a group containing nonabelian simple subgroups H_i such that

$$H_1 \leq H_2 \leq H_3 \leq \dots \quad \text{and} \quad \cup_{n=1}^{\infty} H_n = G.$$

   (a) Prove that G is simple.

   (b) Prove that if $H_n \neq H_{n+1}$ for all n, then G is not finitely generated.

5. Let p be a prime and let P be a nonabelian group of order p^3.

   (a) Prove that the center of P has order p, i.e., that $\#Z(P) = p$.

   (b) Prove that the center of P equals the commutator subgroup of P, i.e., $Z(P) = P'$.

6. Let G be a *solvable* group of order $168 = 2^3 \cdot 3 \cdot 7$. The aim of this exercise is to show that G has a normal Sylow p-subgroup for some prime p. Let M be a minimal normal subgroup of G.

   (a) Show that if M is not a Sylow p-subgroup for any prime p, then $\#M = 2$ or 4. (You may quote without proof any result you need about minimal normal subgroups of solvable groups.)

   (b) Assume that $\#M = 2$ or 4 and let $\overline{G} = G/M$. Prove that $\overline{G}$ has a normal Sylow 7-subgroup.

   (c) Under the same assumptions and notations as (b), let H be the complete preimage in G of the normal Sylow 7-subgroup of $\overline{G}$. Prove that H has a normal Sylow 7-subgroup P, and deduce that P is normal in G.