Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

- Look up your objective scores

  First half of semester : A1 – C8 I will grade Redo HW 1 today/tom

  Second half of semester: D1 – onwards

  Redo HW2

- Redo HW2: undergrads: don't need to do anything

  graduate: do have to compute each integral at least once

Say you want to get a better score on objective

D1: Solve an integral using the definition

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$\gamma(t)$ $a \leq t \leq b$

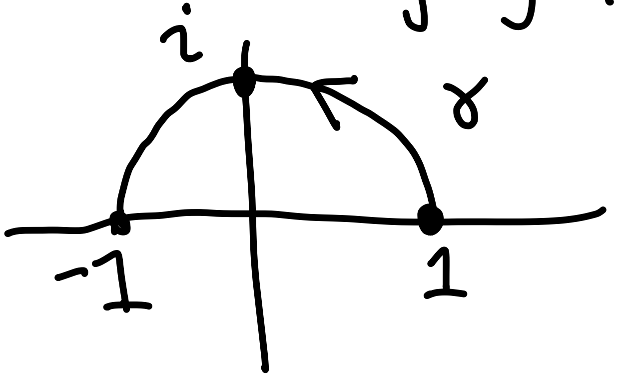
→ choose 2 integrals (a, b, c, d, e) and solve them using the definition

need to use the technique at least once in the problem

Objective D1 ← write this for me please

a) $\int_\gamma z \, dz$

γ half circle from 1 to -1 going through



5 techniques

D1: definition

D2: antiderivative

D3: Cauchy's Theorem

D4: Cauchy's Integral Formula

can also use Cauchy's Generalized Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

$$f^{(k)}(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{k+1}}\, dz$$

E6: $\int_\gamma f(z)\, dz = 2\pi i\, c_{-1}$

coefficient of Laurent series centered at z_0

z_0 inside γ

Goal: Have everything turned in by Thursday Dec 10 at 11:59pm.

Deadline for Redo HW2 is Friday 11:59pm

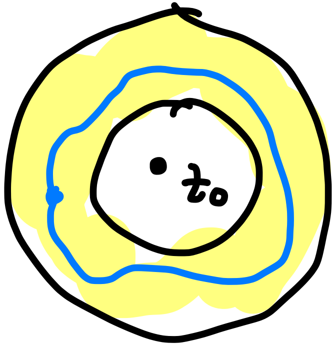
hard deadline, no automatic extension (but you may email/message me to ask for one)

Anyone for integrating?

Objective E6: Use the Laurent series | Residue theorem

Corollary 8.27: f is holomorphic on an annulus

$R_1 < |z-z_0| < R_2$



$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z-z_0)^k$$

γ simple, closed, piecewise smooth in annulus

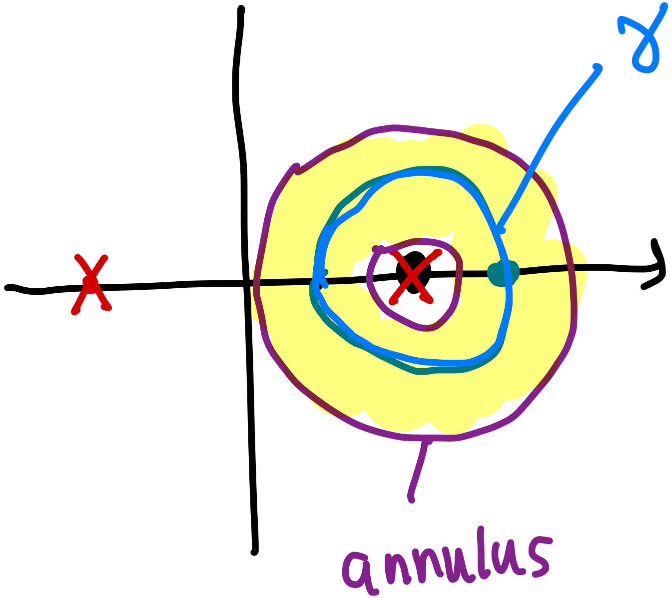
then

$$\int_\gamma f(z)\,dz = 2\pi i \, c_{-1}$$

c) $\int_\gamma \frac{1}{(z-2)^2(z+2)}\,dz$

$f(z) = \frac{1}{(z-2)^2(z+2)}$ hol on $U = \mathbb{C} - \{2, -2\}$

γ circle of radius 1 around $z=2$



$z_0 = 2$

If I can write

$$f(z) = \frac{1}{(z-2)^2(z+2)} = \sum c_k (z-2)^k \quad k \in \mathbb{Z}$$

then $\int_\gamma f(z)\,dz = 2\pi i\, c_{-1}$

Whole problem rests on finding this Laurent series.

Note: Partial fraction decomposition is not the most efficient

$$\frac{1}{(z-2)^2(z+2)} = f(z) = \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{z+2}$$

$$= (z-2)^{-2}\boxed{\frac{1}{z+2}}$$

$$= \underbrace{A(z-2)^{-1}} + \underbrace{B(z-2)^{-2}} + \boxed{\text{Laurent series}}$$

Laurent series centered at $z=2$

find Laurent series centered at $z=2$ for this: look like $b_0 + b_1(z-2) + b_2(z-2)^2 + \ldots$

Focus on $\dfrac{1}{z+2}$ for a bit:

this looks like $\dfrac{1}{1-z} = \sum_{k=0}^{\infty} z^k$

want $\dfrac{1}{1-(z-2)} = \sum_{k=0}^{\infty} (z-2)^k$

$$\frac{1}{z+2} = \frac{1}{2+z}$$

$$= \frac{1}{4+(z-2)}$$

$$= \frac{1}{4\left(1+\frac{z-2}{4}\right)}$$

$$= \frac{1}{4} \frac{1}{1+\frac{z-2}{4}}$$

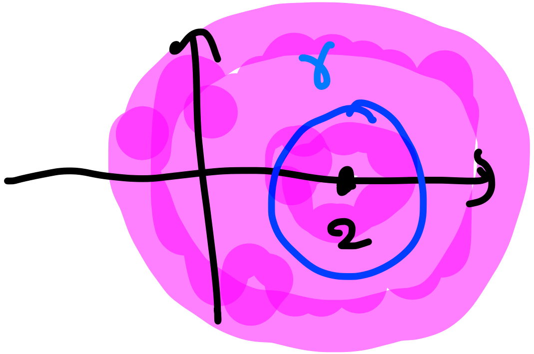
$$= \frac{1}{4} \frac{1}{1-\left(\frac{-(z-2)}{4}\right)}$$

$$= \frac{1}{4} \sum_{k=0}^{\infty} \left[\frac{-(z-2)}{4}\right]^k = \sum_{k=0}^{\infty} \frac{(-1)^k (z-2)^k}{4^{k+1}}$$

this converges if $\left|-\frac{(z-2)}{4}\right| < 1$

$$\frac{|z-2|}{4} < 1, \quad |z-2| < 4$$

$$f(z) = \frac{1}{(z-2)^2(z+2)} = \boxed{(z-2)^{-2}\sum_{k=0}^{\infty}\frac{(-1)^k(z-2)^k}{4^{k+1}}} \quad 0<|z-2|<4$$



Series is valid

$$\int_\gamma f(z)\,dz = 2\pi i\, c_{-1}$$

of this Laurent series

That's all for today!