#3 $|z|=1$ $\boxed{\text{if and only if}}$ $\frac{1}{z}=\bar{z}$

$\boxed{|z|=1 \Rightarrow \frac{1}{z}=\bar{z}}$ & $\boxed{\frac{1}{z}=\bar{z} \Rightarrow |z|=1}$

Let $z=x+iy$. If $|z|=1$, then $\sqrt{x^2+y^2}=1$

or $x^2+y^2=1$. Then

$$\frac{1}{z}=\frac{1}{x+iy}\cdot\frac{(x-iy)}{x-iy}=\frac{x-iy}{x^2-ixy+ixy+y^2}=\frac{x-iy}{x^2+y^2}$$

$$\frac{1}{z} = \ldots\ldots = \frac{x - iy}{x^2 + y^2} = \frac{x - iy}{1} = x - iy = \bar{z}$$

so $\quad \frac{1}{z} = \bar{z}.$

$$\frac{1}{z} = \bar{z} \Rightarrow |z| = 1$$

Let $z = x + iy$, then $\bar{z} = x - iy$ and $\frac{1}{z} = \frac{1}{x+iy}$

To compare $\frac{1}{z}$ and $\bar{z}$, I want to compare real and imaginary parts, so multiply top & bottom of $\frac{1}{z}$ by conjugate

$$\frac{1}{z} = \frac{1}{(x+iy)} \cdot \frac{(x-iy)}{(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$x^2 - i\cancel{x}y + i\cancel{x}y + y^2$$

So $\frac{1}{z} = \bar{z}$ means that

$$\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = x - iy \Rightarrow \begin{array}{l} \frac{x}{x^2+y^2} = x \\ \frac{-y}{x^2+y^2} = -y \end{array}$$

So $x^2+y^2=1$

But $|z|=\sqrt{x^2+y^2}$

so $|z|=\sqrt{x^2+y^2}=\sqrt{1}=1$

so $|z|=1$,