Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Today

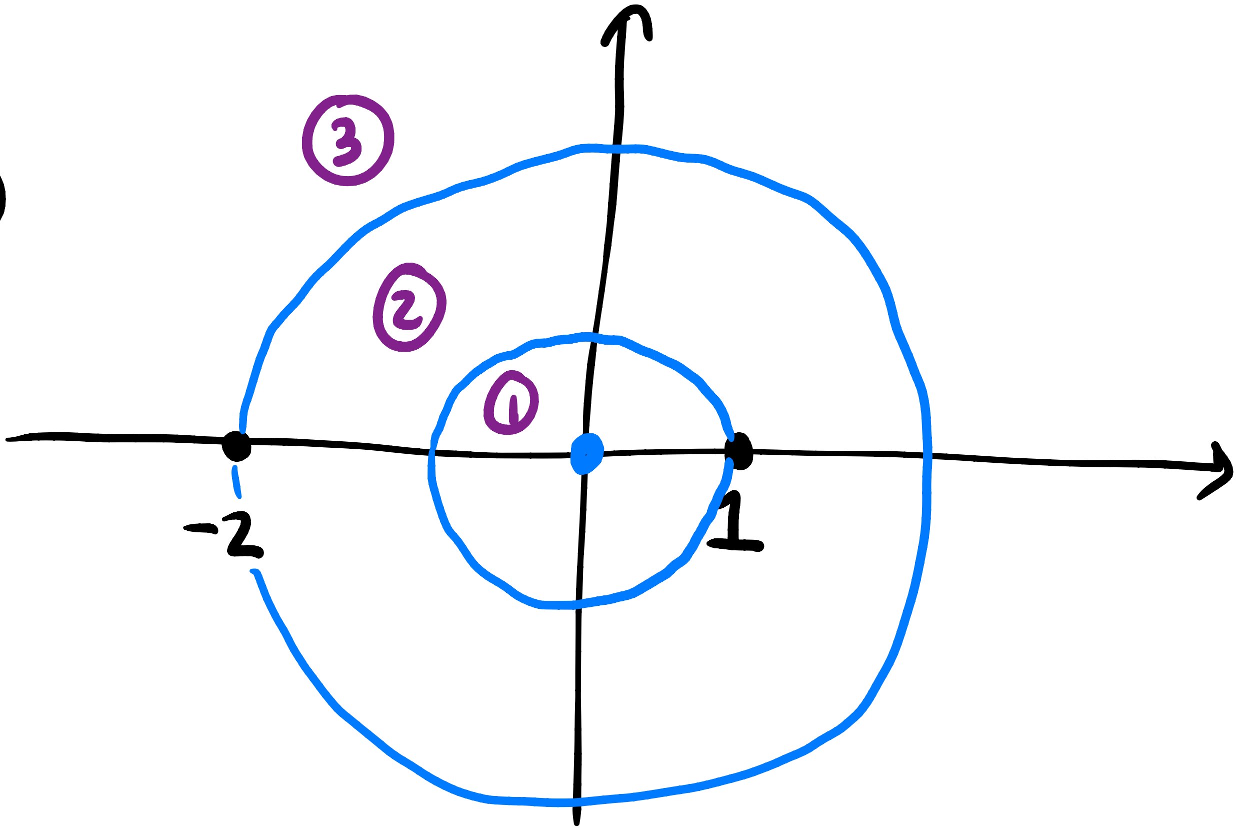
- any questions on HW?
- proof of Theorem 8.24 of BMPS if time

HW9 #2

$$f(z) = \frac{3}{(1-z)(z+2)}$$

holomorphic on

$U = \mathbb{C} - \{1, -2\}$



$$f(z) = \frac{3}{(1-z)(z+2)} = 3 \cdot \frac{1}{1-z} \cdot \frac{1}{z+2}$$

multiplication

Partial fraction decomp

$$= 3\left(\frac{\overset{\cdot(z+2)}{A}}{\underset{(z+2)}{1-z}} + \frac{\overset{\cdot(1-z)}{B}}{\underset{(1-z)}{z+2}}\right) = 3\left(\frac{A(z+2) + B(1-z)}{(1-z)(z+2)}\right)$$

$$\Rightarrow 1 = Az + 2A + B - Bz$$

$$0z + 1 = (A-B)z + (2A+B)$$

$$\Rightarrow A - B = 0 \qquad A = B$$

$$2A + B = 1$$

$$2A + A = 1 \qquad A = B = \frac{1}{3}$$

$$3A = 1$$

$$f(z) = \frac{3}{(1-z)(z+2)} = \frac{\cancel{1/3}\ 1}{1-z} + \frac{\cancel{1/3}\ 1}{z+2}$$

$$\frac{1}{1-z} = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad \text{when } |z|<1$$

Warm-up 9.1

$$\frac{-z}{1-z} = \sum_{-\infty}^{k=0} z^k \qquad |z|>1$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \dots$$

$$\boxed{\begin{aligned} \frac{1}{1-z} &= \frac{-1}{z} \cdot \frac{-z}{1-z} \\ &= \frac{-1}{z} \sum_{-\infty}^{k=0} z^k = -\left(z^{-1} + z^{-2} + \dots\right) \\ &= -\sum_{-\infty}^{k=0} z^{k-1} = -\frac{1}{z} - \frac{1}{z^2} - \dots \end{aligned}}$$

$$\frac{1}{1-z} = \begin{cases} \sum_{k=0}^{\infty} z^k & \text{when } |z|<1 \\ -\sum_{k=1}^{\infty} z^{-k} & \text{when } |z|>1 \end{cases}$$

$$\frac{1}{z+2} = \frac{1}{2+z} = \frac{1}{2\left(1+\frac{z}{2}\right)} = \frac{1}{2\left(1-\left(\frac{-z}{2}\right)\right)}$$

$$\frac{1}{1-z} = \begin{cases} \displaystyle\sum_{k=0}^{\infty} z^{k} & \text{when } |z|<1 \\ \displaystyle -\sum_{k=1}^{\infty} z^{-k} & \text{when } |z|>1 \end{cases}$$

$$\frac{1}{z+2} = \frac{1}{2\left(1-\left(-\frac{z}{2}\right)\right)} = \begin{cases} \displaystyle\frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{z}{2}\right)^{k} & \text{when } \left|-\frac{z}{2}\right|<1 \\ \displaystyle -\frac{1}{2}\sum_{k=1}^{\infty}\left(-\frac{z}{2}\right)^{-k} & \text{when } \left|-\frac{z}{2}\right|>1 \end{cases}$$

$$\frac{1}{z+2} = \begin{cases} \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{-z}{2}\right)^{k} & \text{when } \left|\frac{-z}{2}\right|<1 \\ -\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{-z}{2}\right)^{-k} & \text{when } \left|\frac{-z}{2}\right|>1 \end{cases}$$

$$\left|\frac{-z}{2}\right| = \frac{|-z|}{2} = \frac{|z|}{2}$$

$$= \begin{cases} \frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{2^{k}}\, z^{k} & \text{when } |z|<2 \\ -\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k}\, 2^{k}\, z^{-k} & \text{when } |z|>2 \end{cases}$$

$$f(z) = \frac{1}{1-z} + \frac{1}{z+2}$$

whenever $z \neq 1, -2$

$$\begin{cases} \sum_{k=0}^{\infty} z^k & \text{when } |z|<1 \\ -\sum_{k=1}^{\infty} z^{-k} & \text{when } |z|>1 \end{cases}$$

$$\begin{cases} \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^k & \text{when } |z|<2 \\ -\frac{1}{2}\sum_{k=1}^{\infty} (-1)^k 2^k z^{-k} & \text{when } |z|>2 \end{cases}$$

When $|z|<1$

$$f(z) = \sum_{k=0}^{\infty} \left(1 + \frac{(-1)^k}{2^{k+1}}\right) z^k$$

this is a power series

$$f(z) = \frac{1}{1-z} + \frac{1}{z+2}$$

whenever $z \neq 1, -2$

$$\begin{cases} \sum_{k=0}^{\infty} z^k & \text{when } |z|<1 \\ -\sum_{k=1}^{\infty} z^{-k} & \text{when } |z|>1 \end{cases}$$

$$\begin{cases} \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^k & \text{when } |z|<2 \\ -\frac{1}{2}\sum_{k=1}^{\infty} (-1)^k 2^k z^{-k} & \text{when } |z|>2 \end{cases}$$

when $1<|z|<2$

$$f(z) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^k - \sum_{k=1}^{\infty} z^{-k}$$

Laurent series

$$f(z) = \frac{1}{1-z} + \frac{1}{z+2}$$

whenever $z \neq 1, -2$

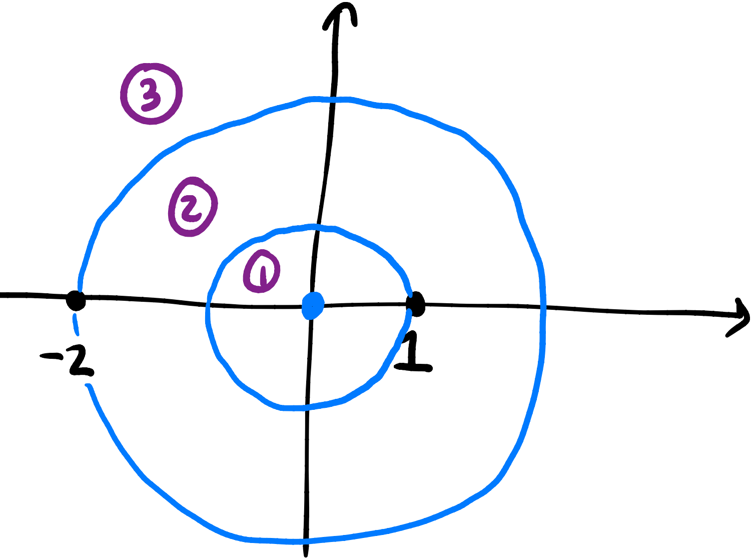
$$\begin{cases} \sum_{k=0}^{\infty} z^k & \text{when } |z|<1 \\ -\sum_{k=1}^{\infty} z^{-k} & \text{when } |z|>1 \end{cases}$$

$$\begin{cases} \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^k & \text{when } |z|<2 \\ -\frac{1}{2}\sum_{k=1}^{\infty} (-1)^k 2^k z^{-k} & \text{when } |z|>2 \end{cases}$$

when $|z|>2$

$$f(z) = -\sum_{k=1}^{\infty} \left(1 + (-1)^k 2^{k-1}\right) z^{-k}$$

Laurent series



$$f(z) = \begin{cases} -\displaystyle\sum_{k=1}^{\infty} \underbrace{\left(1 + (-1)^k 2^{k-1}\right)}_{c_{-k}} z^{-k}, & |z| > 2 \\ \dfrac{1}{2} \displaystyle\sum_{k=0}^{\infty} \dfrac{(-1)^k z^k}{2^k} - \displaystyle\sum_{k=1}^{\infty} z^{-k}, & 1 < |z| < 2 \\ \displaystyle\sum_{k=0}^{\infty} \left(1 + \dfrac{(-1)^k}{2^{k+1}}\right) z^k, & |z| < 1 \end{cases}$$

Theorem 8.24

If f is holomorphic in an annulus, then f has a Laurent series valid at least in that annulus.

Next week: Singularities + Residue theorem

Warm up 10.1 will be posted tonight

The rest will come over the weekend

That's all for today!