Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

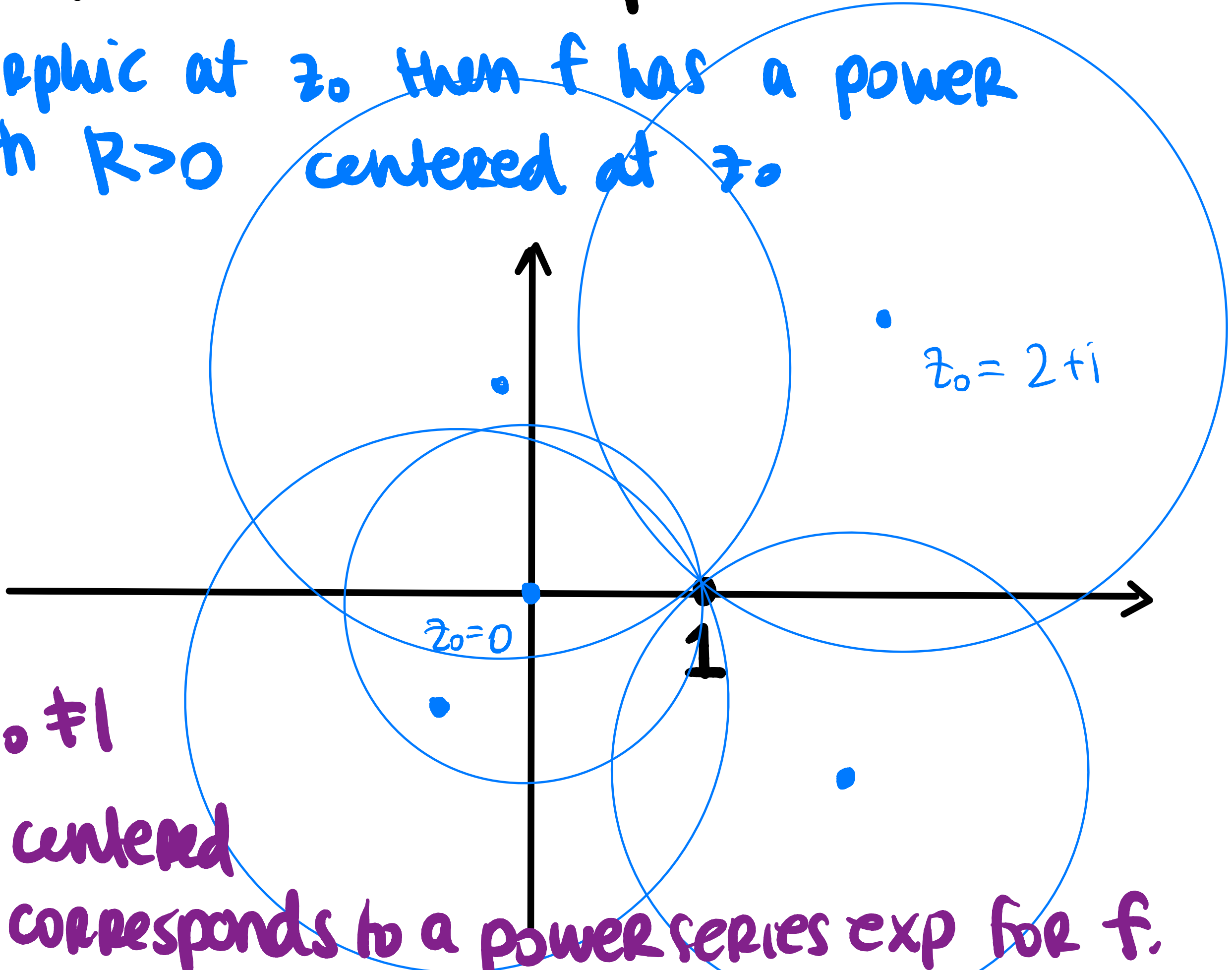
Last week: power series for holomorphic functions

if f is holomorphic at z_0 then f has a power series with $R>0$ centered at z_0

$$\boxed{f(z) = \frac{-z}{1-z}}$$

holomorphic on

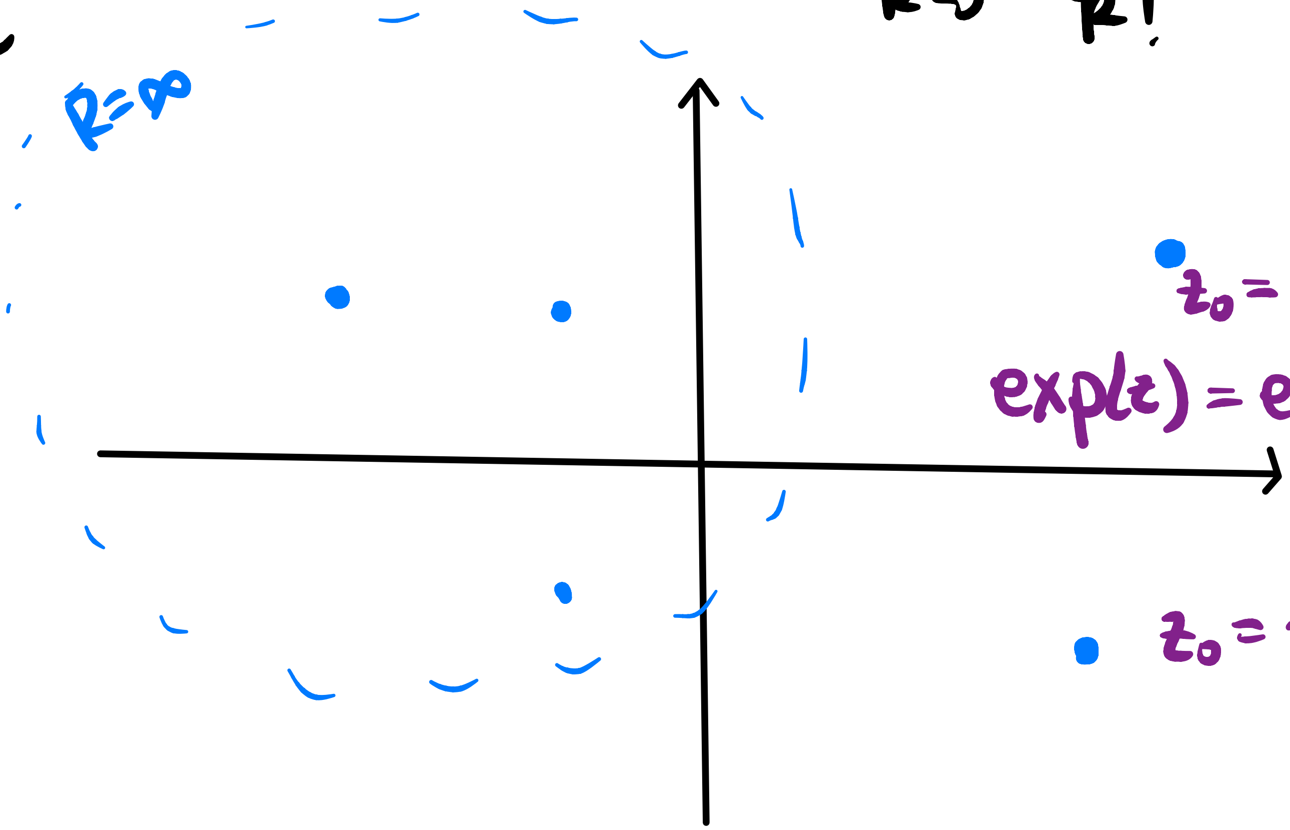
$U = \mathbb{C} - \{1\}$



By moving around $z_0 \neq 1$ cover $\mathbb{C}$ with circles centered at z_0 and each circle corresponds to a power series exp for f.

$$f(z)=\exp(z)=\exp(z_0)\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{k!}\qquad R=\infty$$

$U=\mathbb{C}$



$z_0=3+i$

$$\exp(z)=\exp(3+i)\sum_{k=0}^{\infty}\frac{(z-3-i)^k}{k!}$$

$z_0=2-i$

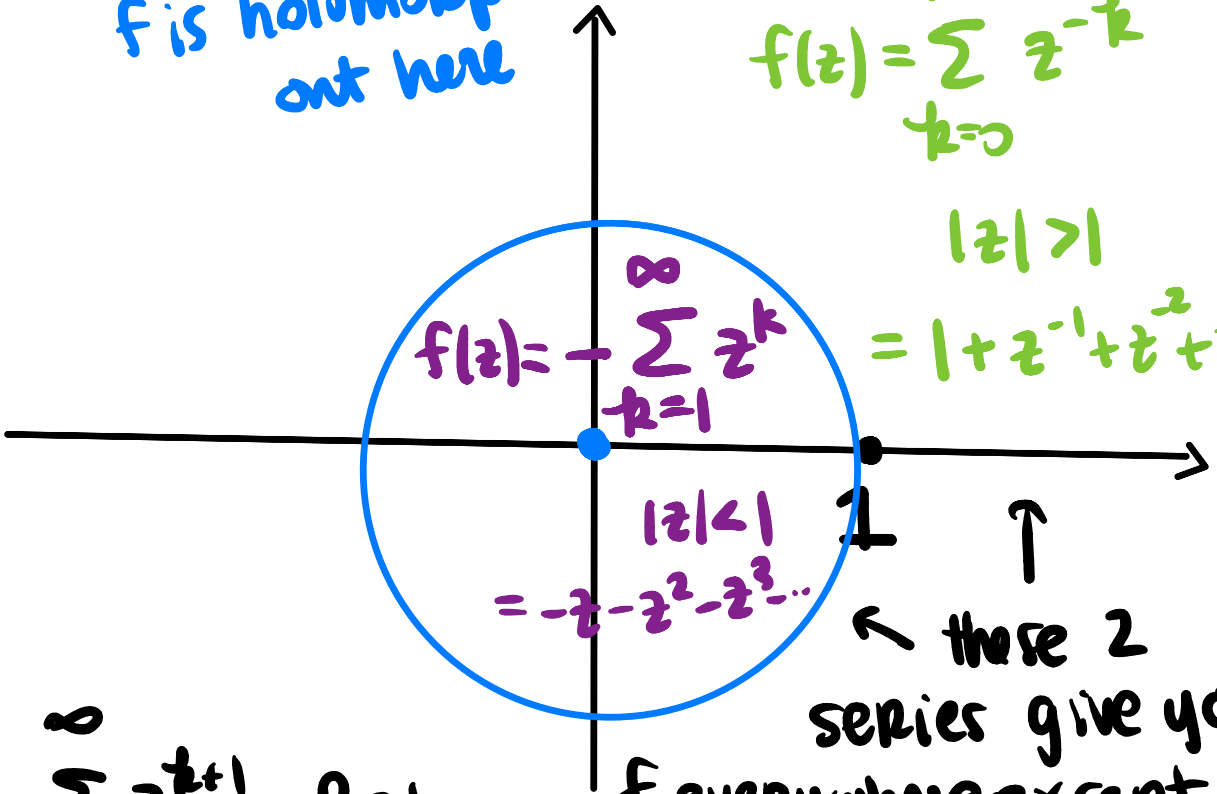
This week: What if we need to stick to a fixed z_0?

$$\boxed{f(z) = \frac{-z}{1-z}}$$

$z_0 = 0$

$$f(z) = -z \cdot \frac{1}{1-z}$$

$$= -z \cdot \sum_{k=0}^{\infty} z^k = -\sum_{k=0}^{\infty} z^{k+1} \qquad R=1$$



$$f(z) = -\sum_{k=1}^{\infty} z^k$$

$|z|<1$

$= -z - z^2 - z^3 - \dots$

$$f(z) = \sum_{k=0}^{\infty} z^{-k}$$

$|z|>1$

$= 1 + z^{-1} + z^{-2} + z^{-3} + \dots$

these 2 series give you f everywhere except maybe $|z|=1$

Section 8.3 of BMPS: Laurent series

Definition: A Laurent series centered at $z_0 \in \mathbb{C}$ is a double series of the form

$$\sum_{k \in \mathbb{Z}} c_k (z - z_0)^k := \sum_{k=0}^{\infty} c_k (z-z_0)^k + \sum_{k=1}^{\infty} c_{-k} (z-z_0)^{-k}$$

$$\|$$

$$\cdots \underbrace{+ c_{-2}(z-z_0)^{-2}}_{\cdots\ k=2} \underbrace{+ c_{-1}(z-z_0)^{-1}}_{k=1} + \underbrace{c_0}_{k=0} + \underbrace{c_1(z-z_0)}_{k=1} + \underbrace{c_2(z-z_0)^2}_{k=2} + \cdots$$

The double series converges iff both series converge.

if and only if

We know that the "power series" half converges in a circle around z_0

$$\sum_{k \in \mathbb{Z}} c_k (z-z_0)^k := \underbrace{\sum_{k=0}^{\infty} c_k (z-z_0)^k}_{\exists R \text{ s.t. this converges for } |z-z_0|<R} + \boxed{\sum_{k=1}^{\infty} c_{-k} (z-z_0)^{-k}}$$

$$\sum_{k\in\mathbb{Z}} c_k (z-z_0)^k := \sum_{k=0}^{\infty} c_k (z-z_0)^k + \sum_{k=1}^{\infty} \overbrace{\boxed{c_{-k}(z-z_0)^{-k}}}^{=:\, a_k}$$

Where does the other half converge?

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{\left|c_{-(k+1)}(z-z_0)^{-(k+1)}\right|}{\left|c_{-k}(z-z_0)^{-k}\right|}$$

$$= \lim_{k\to\infty} \left|\frac{c_{-(k+1)}}{c_{-k}}\right| \frac{1}{|z-z_0|}$$

$$= \frac{1}{|z-z_0|} \lim_{k\to\infty} \left|\frac{c_{-(k+1)}}{c_{-k}}\right|$$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \frac{1}{|z-z_0|} \underbrace{\boxed{\lim_{k\to\infty} \frac{|c_{-(k+1)}|}{|c_{-k}|}}}_{L}$$

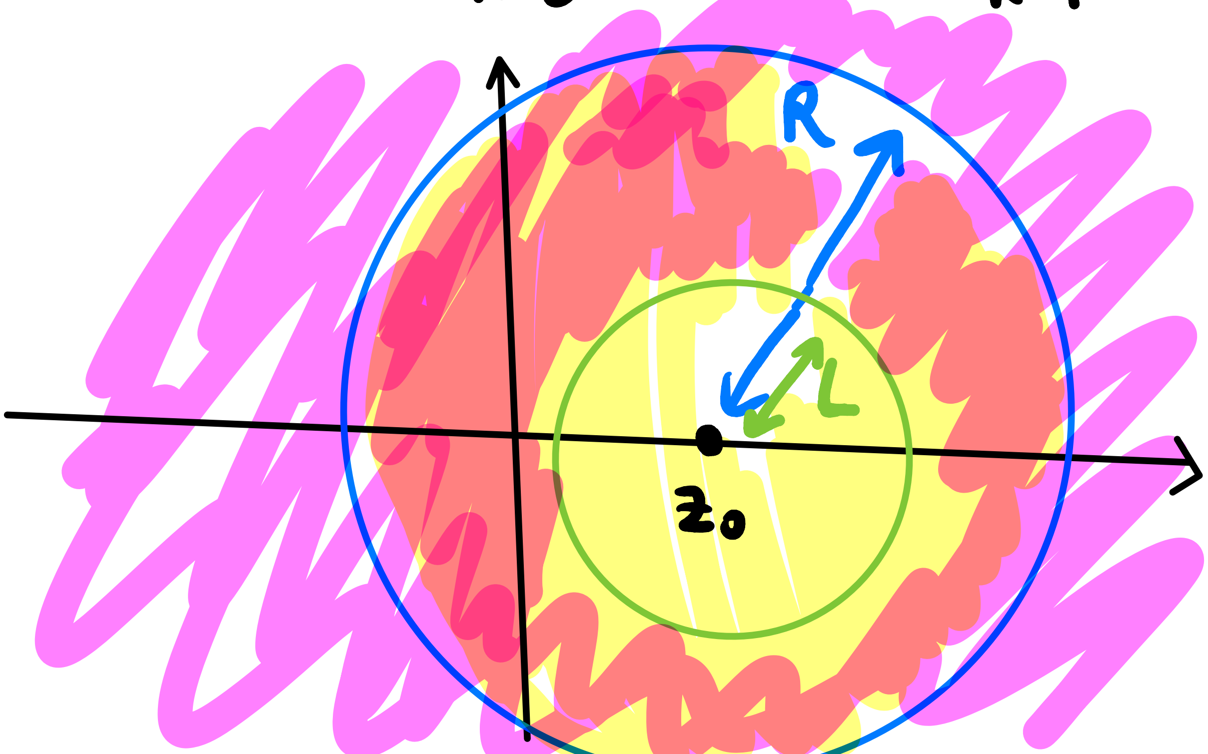
if this limit doesn't exist then whole limit and series diverges

Ratio test says my series converges if

$$\frac{1}{|z-z_0|} L < 1 \Rightarrow L < |z-z_0|$$

my series diverges if $\frac{1}{|z-z_0|} L > 1 \Rightarrow L > |z-z_0|$

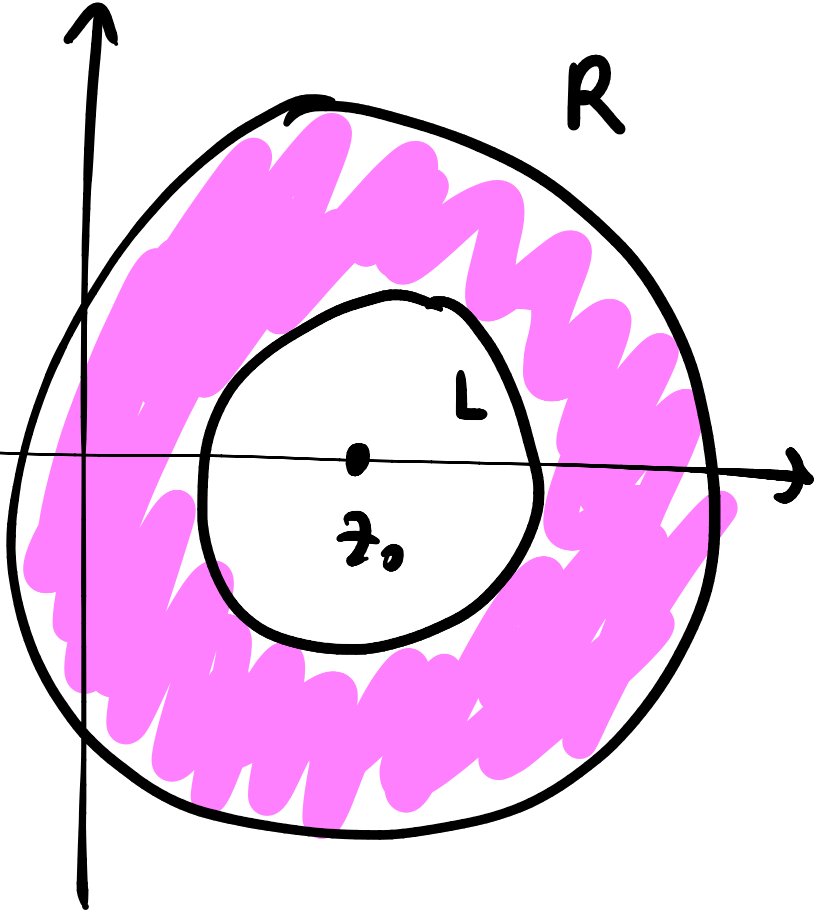
$$\sum_{k\in\mathbb{Z}} c_k (z-z_0)^k := \overbrace{\sum_{k=0}^{\infty} c_k (z-z_0)^k}^{R \quad |z-z_0|<R} + \overbrace{\sum_{k=1}^{\infty} c_{-k} (z-z_0)^{-k}}^{L \quad |z-z_0|>L}$$



If $L < R$

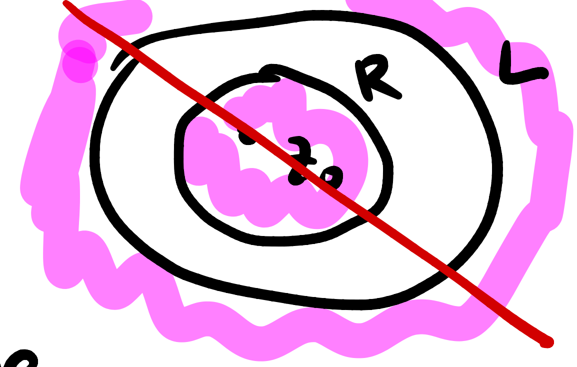
the Laurent series converges in the annulus

$$L < |z - z_0| < R$$

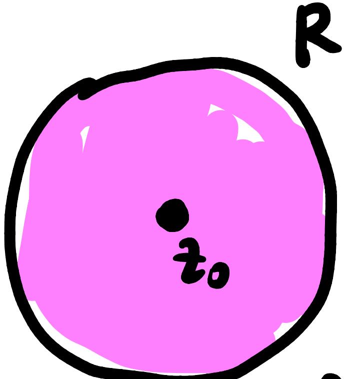


Special cases: if $R<L$

no intersection where both converge so Laurent series does not converge anywhere



if $L=0$



The Laurent series converges at $z=z_0$ iff $c_k=0$ for $k<0$

(this is the case where the Laurent series is a power series)

series converges for $0<|z-z_0|<R$

if $R = \infty$, series converges for $L < |z - z_0|$

There will be warm up 9.2

THAT'S ALL FOR TODAY!