Complex Analysis

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HW8. #1b) We know the power series for $\frac{1}{1-z}$

Goal: $\frac{z^2}{(4-z)^2}$

differentiate:

$$\frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}(1-z)^{-1} = (-1)(1-z)^{-2}(-1) = \frac{1}{(1-z)^2}$$

multiply by $\frac{1}{16}$:

$$\frac{1}{16(1-z)^2} = \frac{1}{4^2(1-z)^2} = \frac{1}{(4(1-z))^2} = \frac{1}{(4-4z)^2}$$

substitute $z \to \frac{z}{4}$

$$\frac{1}{\left(4-\cancel{4}\left(\frac{z}{\cancel{4}}\right)\right)^2} = \frac{1}{(4-z)^2}$$

multiply by z^2

$$\frac{z^2}{(4-z)^2}$$

Steps

1. differentiate
2. multiply by $\frac{1}{16}$
3. substitute $z \to \frac{z}{4}$
4. multiply by z^2

Goal: $\frac{z^2}{(4-z)^2}$

Start with $\sum_{k=0}^{\infty} z^k$

differentiate:

$$\frac{d}{dz} \sum_{k=0}^{\infty} z^k = \sum_{k=1}^{\infty} k z^{k-1}$$

$$\frac{d}{dz}\sum_{k=0}^{\infty} z^k = \frac{d}{dz}\left(1 + z + z^2 + z^3 + \cdots\right)$$

$$= 0 + 1 + 2z + 3z^2 + \cdots$$

$$\Vert$$

$$\sum_{k=0}^{\infty} k z^{k-1} = \cancelto{0}{0\cdot\frac{1}{z}} + \cancelto{1}{1\cdot z^0} + 2z^1 + \cdots$$

$$= \sum_{k=1}^{\infty} k z^{k-1}$$

multiply by $\frac{1}{16}$

$$\frac{1}{16}\sum_{k=1}^{\infty} k z^{k-1} = \sum_{k=1}^{\infty} \frac{k z^{k-1}}{16}$$

Steps $R=1$

1. differentiate $R=1$
2. multiply by $\frac{1}{16}$ $R=1$
3. substitute $|z|<1$, $z \to \frac{z}{4}$, $\left|\frac{z}{4}\right|<1$; $R=4$, $|z|<4$
4. multiply by z^2; $R=4$

substitute $z \to \frac{z}{4}$

$$\sum_{k=1}^{\infty} \frac{k}{16}\left(\frac{z}{4}\right)^{k-1} = \sum_{k=1}^{\infty} \frac{k}{16\cdot 4^{k-1}} z^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{k}{4^2\cdot 4^{k-1}} z^{k-1} = \sum_{k=1}^{\infty} \frac{k}{4^{k+1}} z^{k-1}$$

multiply by z^2

$$z^2 \cdot \sum_{k=1}^{\infty} \frac{k}{4^{k+1}} z^{k-1} = \sum_{k=1}^{\infty} \frac{k}{4^{k+1}} z^2 \cdot z^{k-1}$$

$$= \sum_{k=1}^{\infty} \frac{k}{4^{k+1}} z^{k+1}$$

optional re-indexing

$k=1 \qquad k=2 \qquad k=3$

$$\sum_{k=1}^{\infty} \frac{k}{4^{k+1}} z^{k+1} = \frac{1}{4^2} z^2 + \frac{2}{4^3} z^3 + \frac{3}{4^4} z^4 + \ldots$$

$k-1=1$

substitute

$k \to k-1$

$k=2 \qquad k=3 \qquad k=4$

$k-1 \qquad k \qquad k-1 \qquad k \qquad k-1 \qquad k$

$$= \frac{1}{4^2} z^2 + \frac{2}{4^3} z^3 + \frac{3}{4^4} z^4 + \ldots$$

$k \qquad k \qquad k$

$$= \sum_{k=2}^{\infty} \frac{(k-1) z^k}{4^k}$$

$$\frac{z^2}{(4-z)^2} = \sum_{k=2}^{\infty} \frac{(k-1)z^k}{4^k}$$

Radius of convergence : Ratio test

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left|\frac{k z^{k+1}}{4^{k+1}}\right| \div \left|\frac{(k-1)z^k}{4^k}\right|$$

$$= \lim_{k\to\infty} \left|\frac{k z^{k+1}}{4^{k+1}} \cdot \frac{4^k}{(k-1)z^k}\right|$$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left| \frac{k\, \cancelto{z}{z^{k+1}}}{\cancelto{4}{4^{k+1}}} \cdot \frac{\cancel{4^k}}{(k-1)\cancel{z^k}} \right|$$

$$= \lim_{k\to\infty} \frac{k}{4(k-1)} |z|$$

$$= \lim_{k\to\infty} \frac{1}{4\left(1-\cancelto{0}{\frac{1}{k}}\right)} |z| = \frac{1}{4} |z|$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{4}|z|$$

$z_0 = 0$

Series converges if $\frac{1}{4}|z| < 1$, $|z| < 4$

diverges if $\frac{1}{4}|z| > 1$, $|z| > 4$

R is the number s.t. converges if $|z - z_0| < R$

diverges if $|z - z_0| > R$

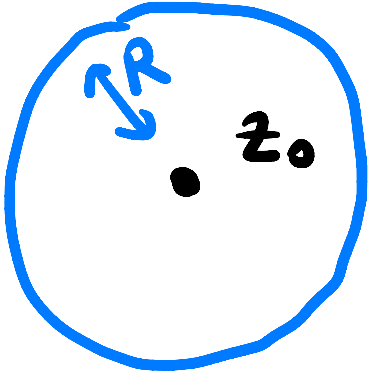
$\Rightarrow R = 4$

HW8 #3

Definition: f is analytic at $z = z_0$ if there is $R > 0$ such that

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$$

for all z in $|z - z_0| < R$



Big theorem:

If f is a complex function ($f: \mathcal{U} \to \mathbb{C}$, $\mathcal{U}$ open in $\mathbb{C}$) then f is holomorphic at $z = z_0$ if and only if f is analytic at $z = z_0$.

f has a power series in a ball around z_0

f' exists in a ball around z_0

In real analysis

analytic $\Rightarrow$ differentiable

$\nLeftarrow$

all derivatives exist + the Taylor series converges

analytic $\Leftrightarrow$ holomorphic

differentiable in a neighborhood of z_0

$f(z) = \exp(z)$ analytic at $z=0$ $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$ $R=\infty$ (Ratio Test)

Let $z_0 \in \mathbb{C}$ <u>if</u> f has a power series centered at z_0, it's the Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

← does this converge around z_0?

For arbitrary $z_0 \in \mathbb{C}$ and $f(z) = \exp(z)$,

$$\boxed{\text{what is } \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k}$$

and does it converge in a neighborhood of z_0?

If $f(z) = \exp(z)$, then $f^{(k)}(z) = \exp(z)$

so $f^{(k)}(z_0) = \exp(z_0)$

The Taylor series of $\exp(z)$ centered at $z = z_0$ is

$$\sum_{k=0}^{\infty} \frac{\exp(z_0)}{k!} (z - z_0)^k$$

$$= \exp(z_0) \sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!}$$

What is R for this series?

* Ratio test
* $R = \infty$

That's all for today!