Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Plan: quick recap

HW 8 #2 d) e)

more theory

#1

Last time

- power series $\sum_{k=0}^{\infty} c_k (z-z_0)^k = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots$

- radius of convergence

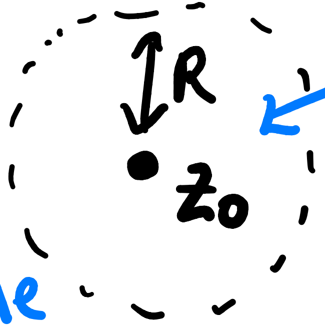
↳ the infinite sum "makes sense" the sums $\sum_{k=0}^{N} c_k (z-z_0)^k$ approach a limit

Not convergent $\begin{cases} 1+(-1)+1+(-1)+\ldots \\ 1+2+3+4+5+\ldots \end{cases}$

Theorem 7.31

There exists $R \in \mathbb{R}_{\geq 0}$, or $R = \infty$ s.t. the region of convergence looks like

does NOT converge outside the circle



converge in the circle

If $R=0$ then converges only at $z=z_0$

(the sum is $c_0+0+0\ldots$)

$R=\infty$ the sum converges for all values of z we may plug in.

Theorem 8.1

if $R>0$

In the region where a power series converges, it defines a holomorphic function.

In that case ($R>0$, and we are inside the region of convergence, which is also the region where

$$f(z)=\sum_{k=0}^{\infty} c_k (z-z_0)^k$$

is holomorphic) then

$$f^{(k)}(z_0) = k!\, c_k$$

kth derivative

evaluated at z_0

Note that this is one way to get a power series for a function #3

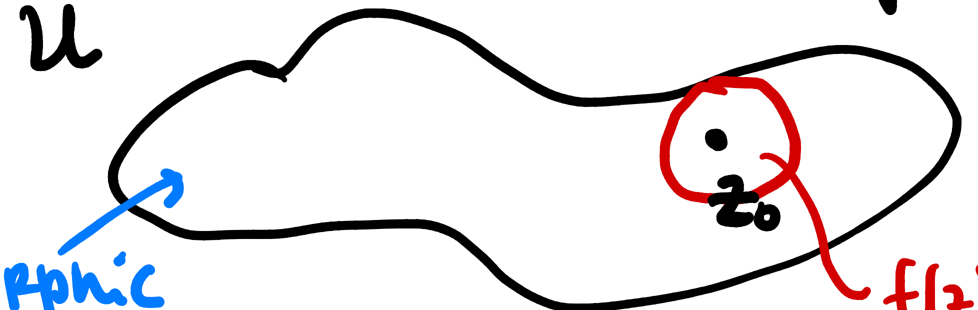
Right now: power series converges $\Rightarrow$ holomorphic

Theorem 8.8 of BMPS

If f is holomorphic in $\{z \in \mathbb{C} : |z - z_0| < R\}$ then f has a power series expansion centered at z_0 with radius of convergence $\geq R$

Corollary 8.9

If f is holomorphic on an open set $\mathcal{U}$, with $z_0 \in \mathcal{U}$ the power series of f centered at z_0 has radius of convergence at least as big as the distance between z_0 and the boundary of $\mathcal{U}$.



because $\mathcal{U}$ open there is a ball around z_0 where f is hol

f is holomorphic

$$f(z) = \sum c_k (z - z_0)^k$$

#2 of HW $f(z) = \frac{1}{z^2+1}$ holomorphic if $z^2+1 \neq 0$

$z^2+1=0$

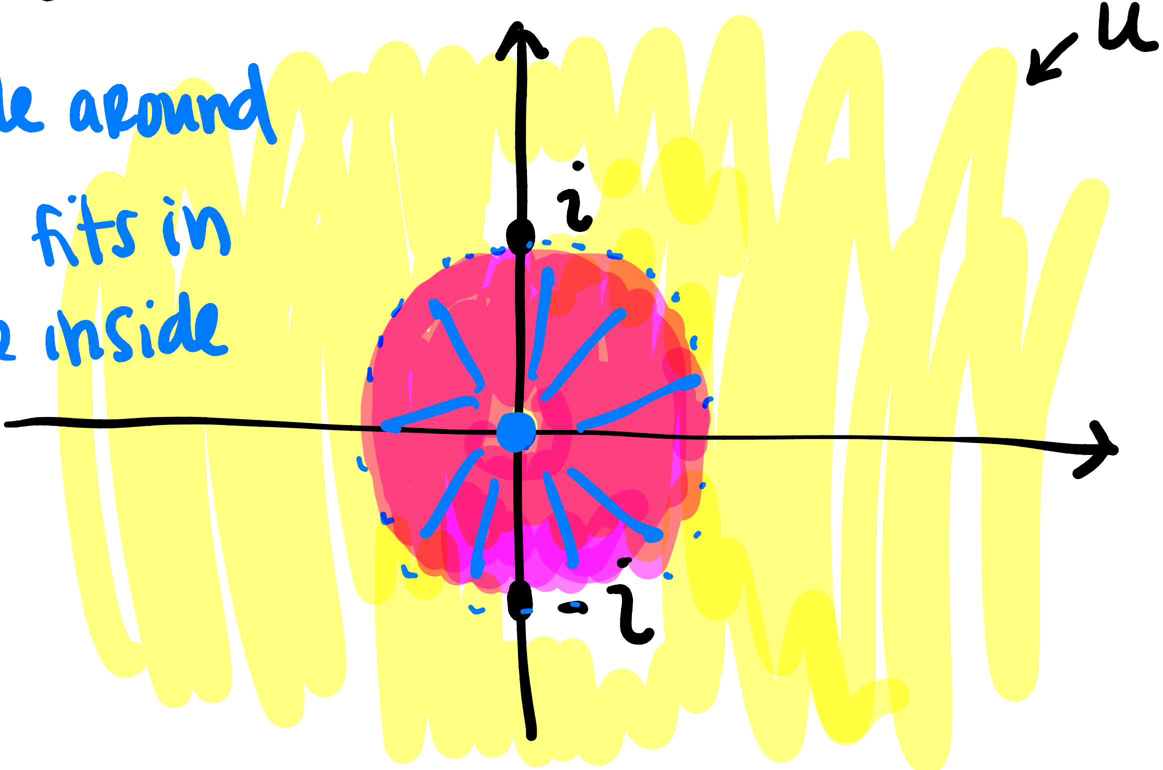
$(z-i)(z+i)=0$

$\Rightarrow z = \pm i$

$$\boxed{U = \mathbb{C} - \{i, -i\}}$$ c)

d) What is the distance between 0 and the "edge of U"

largest circle around $z_0 = 0$ that fits in U is the inside of the unit circle



$R = 1$

Circling back: How do we get a power series expansion for a function?

2 techniques

- using the formula $c_k = \dfrac{f^{(k)}(z_0)}{k!}$

main example $f(z) = \exp(z)$

then $f^{(k)}(z) = \exp(z)$

- modifying known power series

- $$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

centered at $z_0 = 0$

$c_k = 1 \quad \forall k$

$z = z - 0$

geometric series

- $\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ centered at $z_0 = 0$

$z = z - 0$

$c_k = \frac{f^{(k)}(0)}{k!}$

doing this for general z_0 is #3

$= \frac{\exp(0)}{k!} = \frac{1}{k!}$

$z_0 = 0$

- $\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

$$z_0 = 0$$

Back to #2

$$g(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad R=1$$

check using Ratio test

$$f(z) = \frac{1}{z^2+1} = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = g(-z^2)$$

$$= \sum_{k=0}^{\infty} (-z^2)^k = \sum_{k=0}^{\infty} (-1)^k z^{2k}$$

a)

b) want radius of convergence of

$$\sum_{k=0}^{\infty} (-z^2)^k = \sum_{k=0}^{\infty} (-1)^k z^{2k}$$

2 ways to get this, will do both.

Straightforward way: Ratio Test

Ratio test on $\sum_{k=0}^{\infty} (-1)^k z^{2k}$ $\qquad a_k = (-1)^k z^{2k}$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|(-1)^{k+1} z^{2(k+1)}|}{|(-1)^k z^{2k}|}$$

$$= \lim_{k\to\infty} \frac{|z|^{2(k+1)}}{|z|^{2k}} = \lim_{k\to\infty} \frac{|z|^{\cancel{2k}+2}}{|z|^{\cancel{2k}}}$$

$$= \lim_{k\to\infty} |z|^2 = |z|^2$$

Ratio test says this converges if $|z|^2 < 1$

diverges if $|z|^2 > 1$

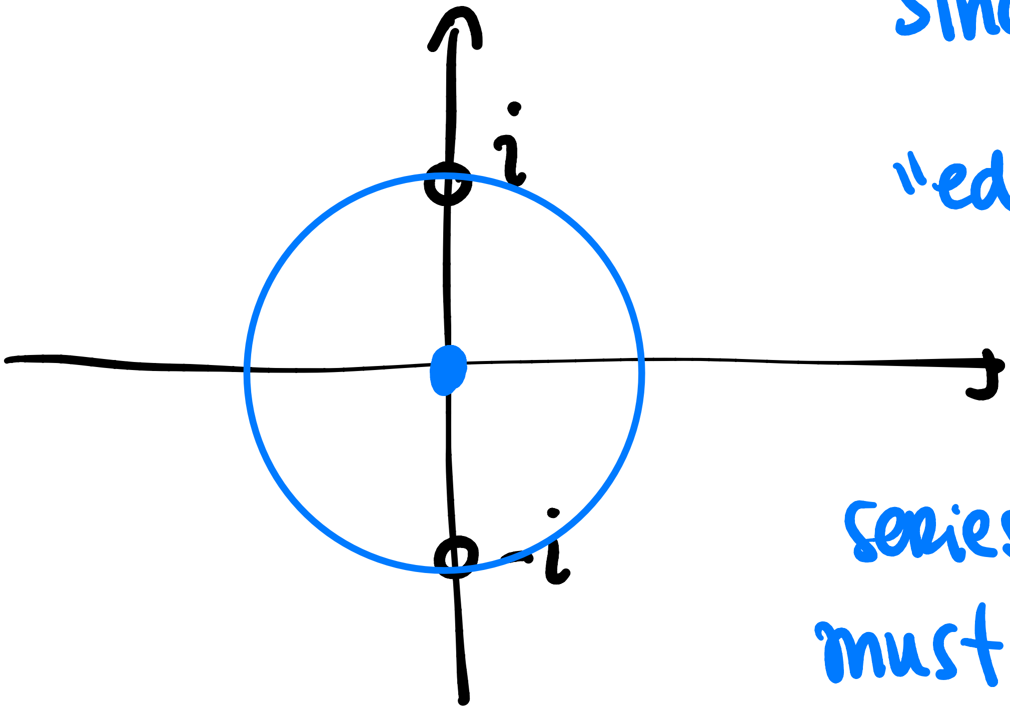
$|z|^2 < 1$ iff $|z| < 1$

$|z|^2 > 1$ iff $|z| > 1$

b) $R = 1$

Other technique on Friday

d) $f(z) = \dfrac{1}{z^2+1} = \sum_{k=0}^{\infty} (-1)^k z^{2k}$ $\quad R=1$



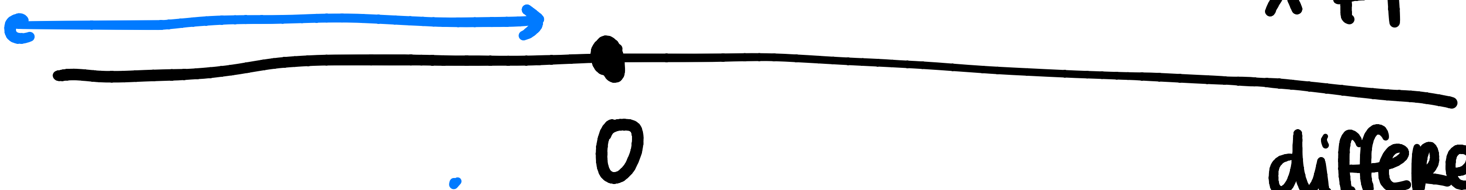
Since the distance between $z_0 = 0$ and the "edge" of $\mathcal{U}$ is 1

Corollary 8.9 says that the power series of f centered at 0 must have $R \geq 1$

e) think only about $\mathbb{R}$

$$\frac{1}{x^2+1}$$

distance between 0 and edge



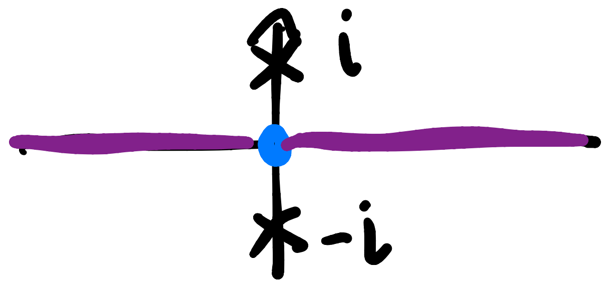
of U is infinite

differentiable everywhere

$U = \mathbb{R}$

If we applied Cor 8.9 (incorrectly since we are in $\mathbb{R}$ not $\mathbb{C}$) we would expect $R \geq \infty$ but in reality $R = 1$

This doesn't contradict Corollary 8.9 because Corollary 8.9 asks for the distance in <u>all</u> directions in $\mathbb{C}$ but in $\mathbb{R}$ we are only looking at the distance to the left and right



THAT'S ALL FOR TODAY!

Friday

- recap
- little bit more lecturing
- HW questions #1b)