Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Is anyone using the Knopp book?

A. Yes

B. No

→ Except if I hear privately from someone who loves it, I will stop giving references to Knopp.

What class experience do you like better?

A. Like earlier when class more active (problems to solve, etc)

B. Like now with more lecturing

C. Other

(Temporary) Wrap up of integration

Recap: $\int_\gamma f(z)\,dz \qquad \gamma(t) \quad a \le t \le b$

① Definition $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

Pros: always works/available

Cons: can be very/too complicated

② Fundamental Thm of Calculus

If f has an antiderivative F, and F is holomorphic on the path γ

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$$

Pros: super easy

Cons: not always available
must find F

③ Cauchy's Theorem

f holomorphic on $\mathcal{U}$, $\gamma_1 \sim_{\mathcal{U}} \gamma_2$

hard path

easier path

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

PROS: this can save you from a hard integration

Cons: still have to compute $\int_{\gamma_2} f(z)\,dz$

④ Cauchy's Integral Formula

f is holomorphic on U, γ and its interior $\subseteq U$

γ closed, simple

+ly oriented

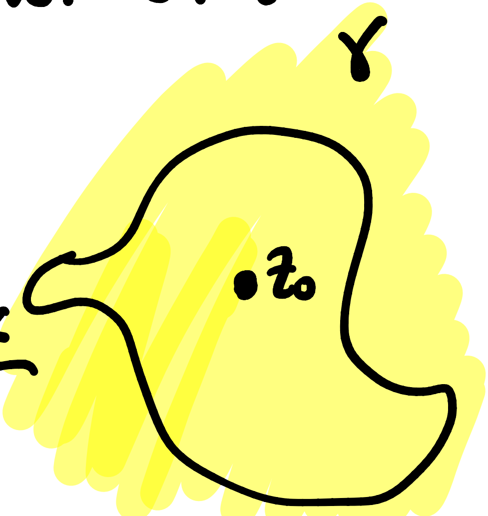
z_0 interior of γ

$$\int_\gamma \boxed{\frac{f(z)}{z-z_0}}\, dz = 2\pi i\, f(z_0)$$

$\parallel$

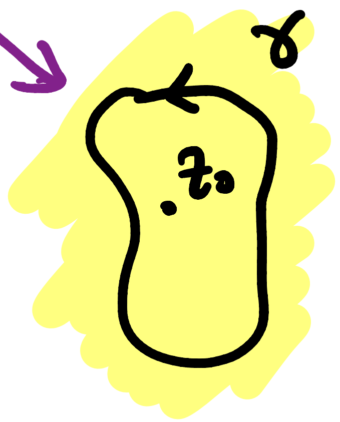
$\int_\gamma g(z)\, dz$

f holomorphic



Next steps: BMPS

f holomorphic



CIF: $$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

Next: $$f'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^2}\, dz$$ ← there's f'

$$f''(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^3}\, dz$$

this tells you that if f is holomorphic, so is f'

This is HUGE!

If f is holomorphic (we really need differentiable on an <u>open set</u>, not just at a point) then f' is holomorphic.

By induction: f is infinitely differentiable and all those derivatives are holomorphic.

Next up: Taylor series

Definition: <u>Power series</u> centered at z_0 is an expression of the form

$$f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \dots,$$

$c_k \in \mathbb{C}$

Examples

- $z_0 = 2$ $\quad c_0 = 1 \quad c_1 = 1+i \quad c_2 = i \quad c_3 = \frac{\sqrt{2}}{2}$

  $c_4 = c_5 = \ldots = 0$

$$f(z) = 1 + (1+i)(z-2) + i(z-2)^2 + \frac{\sqrt{2}}{2}(z-2)^3$$

this power series is a polynomial because it ends

- $z_0 = 0 \quad c_k = 1 \quad k = 0, 1, 2 \ldots$

$$f(z) = \underset{k=0}{1} + \underset{k=1}{z} + \underset{k=2}{z^2} + \underset{k=3}{z^3} + \ldots = \sum_{k=0}^{\infty} z^k$$

Theorem 7.31 of BMPS

Given a power series $\sum_{k=0}^{\infty} c_k (z-z_0)^k$

there exists $R \geq 0$ (R is a real number OR ∞)

such that

(a) If $|z-z_0| < R$ then the series converges

(b) If $|z-z_0| > R$ then the series diverges

Let's talk about conve encerg

A finite sum is always defined

An infinite sum might not be

Example: $1+1+1+1+1+\ldots$ ← sum is not a #, it is infinite

$1+(-1)+1+(-1)+1+(-1)+\ldots$

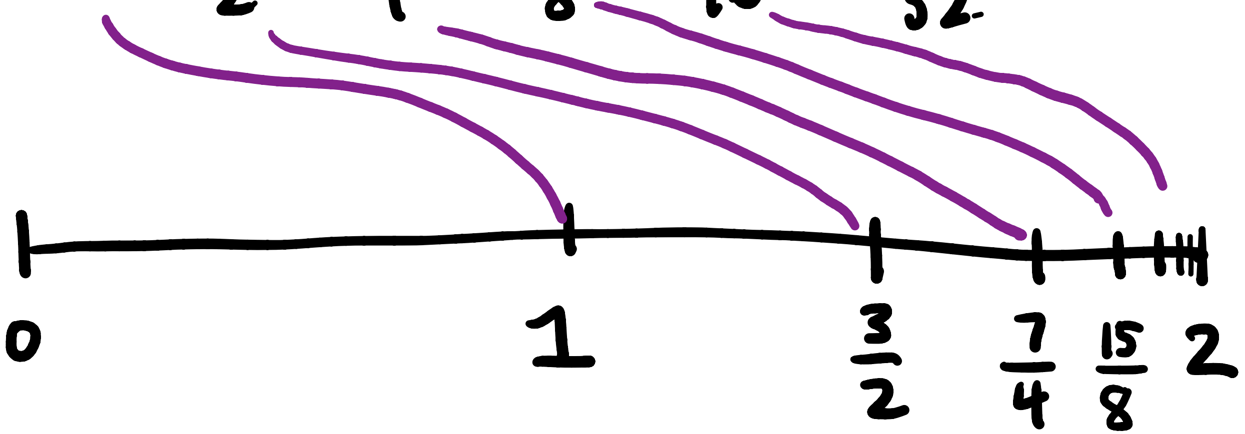
this doesn't get big, but it never settles

the infinite sum doesn't exist

DIVERGES

Sometimes the infinite sum does make sense!

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\ldots = 2$$



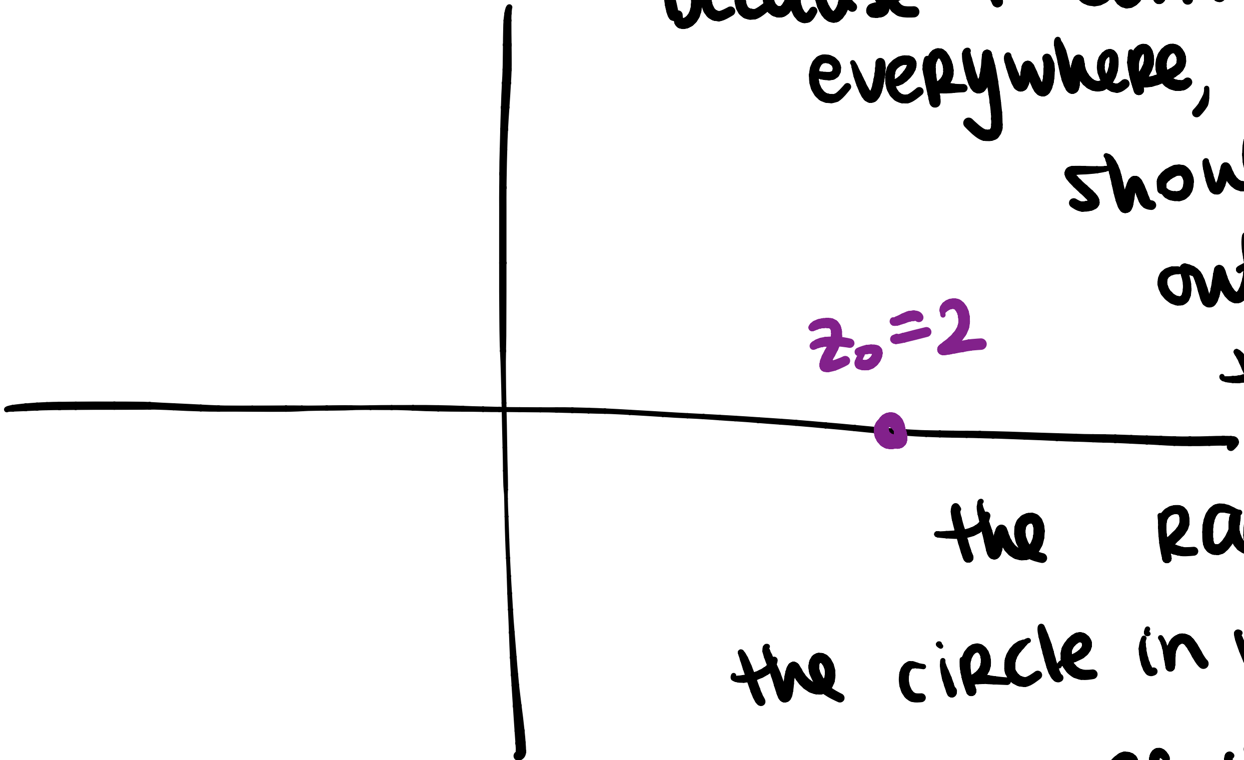
you can get as close as you want to 2 with this sum by adding more terms.

this series CONVERGES to 2

Back to our examples:

$$f(z) = 1 + (1+i)(z-\underset{z_0=2}{2}) + i(z-2)^2 + \frac{\sqrt{2}}{2}(z-2)^3$$

If we plug in values for z, we'll get a sum (here it's finite) A finite sum is always defined, f always gives a finite sum, so any $z \in \mathbb{C}$ is ok to plug in, the sum will converge.



Because f converges everywhere, and it should diverge outside of the circle, the radius of the circle in which f converges in infinite

$$R=\infty$$

No warm up for Wednesday

No more problems on HW 8

That's all for today!