Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

HW 6 is posted

videos + sections in the books are posted

Metacognition

2 objectives : 2 techniques to evaluate

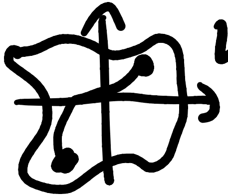
$$\int_\gamma f(z)\,dz$$

Recap: $\mathcal{U}$ a region in $\mathbb{C}$ (open + connected)

f continuous on $\mathcal{U}$

γ a differentiable (smooth) contour $\subseteq \mathcal{U}$

$\gamma : [a,b] \to \mathbb{C}$

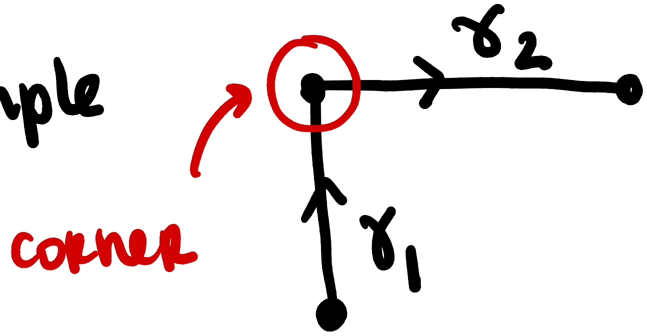


Define the integral of f along γ

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

Bowman $\oint_\gamma f(z)dz$

Example



γ_1 differentiable

γ_2 differentiable

$\gamma_1 + \gamma_2$ is not differentiable

For γ piecewise differentiable, $\gamma = \gamma_1 + \gamma_2$, γ_1, γ_2 differentiable, define

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

Task 167 For each $\gamma(t) = Re^{it}$ $0 \le t \le 2\pi$

$\gamma'(t) = Rie^{it}$ derivative

Calculus

$$\frac{d}{dt} e^{at} = e^{at} \cdot a$$

chain rule $\frac{de^x}{dx} = e^x$ $\frac{d}{dt} at = a$

think u-sub

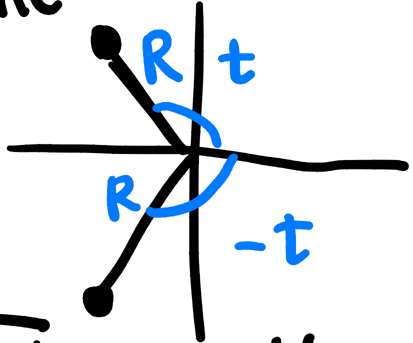
$$z = \gamma(t) = Re^{it}$$

$$dz = \gamma'(t)\,dt = Rie^{it}\,dt$$

① $$\int_\gamma \bar{z}\, dz = \int_0^{2\pi} \overline{(Re^{it})}\, Rie^{it}\, dt$$

$$= \int_0^{2\pi} Re^{-it}\, Rie^{it}\, dt$$

Re^{it}



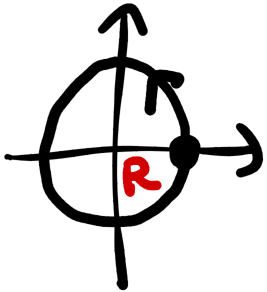
$$\overline{Re^{it}} = Re^{-it}$$

① $\int_\gamma \bar{z}\,dz = \int_0^{2\pi} \overline{(Re^{it})}\, Rie^{it}\,dt$

$= \int_0^{2\pi} Re^{-it} Rie^{it}\,dt$

$e^{-it}e^{it} = e^{-it+it} = e^0 = 1$

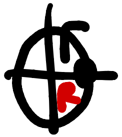
$= R^2 i \int_0^{2\pi} dt$



$= R^2 i\ t\Big|_0^{2\pi}$

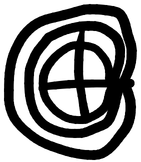
$= R^2 i\,(2\pi - 0) = 2\pi R^2 i$

② $\int_\gamma z^2\,dz$



$\gamma(t) = Re^{it} \quad 0 \le t \le 2\pi$

$\gamma'(t) = Rie^{it}$



$$= \int_0^{2\pi} (Re^{it})^2\, Rie^{it}\, dt$$

$z = \gamma(t)$

$e^{2it}e^{it} = e^{2it+it} = e^{3it}$

$$= \int_0^{2\pi} R^2 e^{2it}\, Ri\, e^{it}\, dt$$

Calculus

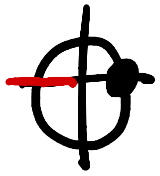
$\int e^{at}\,dt = \frac{e^{at}}{a} + C$

$$= R^3 i \int_0^{2\pi} e^{3it}\, dt$$

$a = 3i$

$$= R^3 \cancel{i}\, \frac{e^{3it}}{3\cancel{i}}\Big|_0^{2\pi} = \frac{R^3}{3}\left[e^{6i\pi} - e^0\right] = \frac{R^3}{3}[1-1] = 0$$

③ $$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\cancel{R}\cancel{e^{it}}}\,\cancel{R}i\cancel{e^{it}}\,dt$$



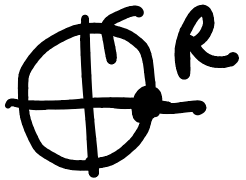
$$= \int_0^{2\pi} i\,dt$$

$$= it\Big|_0^{2\pi} = i\left[2\pi - 0\right] = 2\pi i$$

Calculus $\int \frac{1}{x}\,dx = \ln x + C$

Let's do $\int_\gamma z^2 dz$ $\quad \gamma(t) = Re^{it} \quad 0 \le t \le 2\pi$

using an antiderivative instead



$f(z) = z^2$ looking for $F(z)$ such that $F'(z) = f(z)$

Want F with $F'(z) = z^2$. This is $F(z) = \frac{z^3}{3} + C$

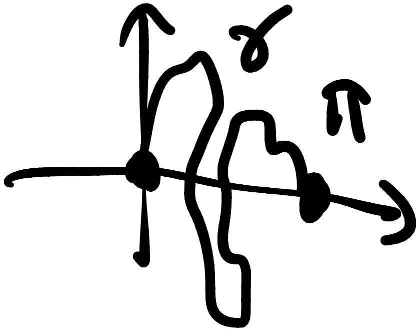
$$\gamma(2\pi) = Re^{2\pi i} = R$$
$$\gamma(0) = Re^{0} = R$$

$$\int_\gamma z^2 dz = F(\gamma(2\pi)) - F(\gamma(0))$$
$$= \frac{R^3}{3} - \frac{R^3}{3} = 0$$

Task 174

γ any contour from 0 to π

$$\int_\gamma e^{iz}\,dz$$



$$= \frac{e^{iz}}{i}\Bigg|_{z=0}^{z=\pi}$$

$$= \frac{e^{i\pi}}{i} - \frac{e^{0}}{i} = \frac{-1}{i} - \frac{1}{i} = \frac{-2}{i}\,\frac{-i}{-i} = \frac{2i}{1} = 2i$$

That's all for today!