Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Questions?

Difference between $\boxed{\exp(z)}$ and $\boxed{e^z}$?

not ambiguous

complex exponentiation
multivalued

Yes and no!

$$\exp(z) = \underbrace{\boxed{e^x}}_{\text{Euler's number}}^{\in\mathbb{R}} \cdot e^{iy} = e^x(\cos y + i \sin y)$$

$z \in \mathbb{C}$

$z = x + iy$

Euler's number

$e \in \mathbb{R}$ $\quad e = 2.\ldots\ldots = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

exp is a complex function, well-defined

(it takes a cx # & outputs a cx #)

$$\exp : \mathbb{C} \to \mathbb{C} \qquad \text{entire}$$

e^z is multivalued i.e. it can be given many values that all make sense

including $e^z = \exp(z)$

this is the principal value of e^z

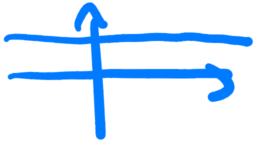
In the chat: HW5 #1d) do it in general on Wednesday

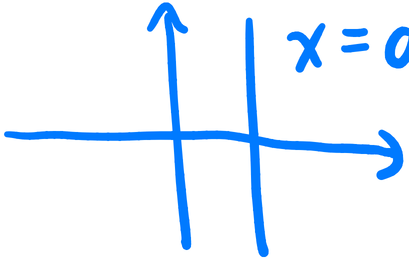
today $n=2$ $\quad f(z)=z^2$

Warm up 5.1 #1

a) $(x+iy)^2 = (\qquad) + i(\qquad)$

$x,y \qquad\qquad x,y$

c)  $y=b \rightsquigarrow z=x+ib$ $\quad x$ varies $\quad b=0,\pm1,\pm2$

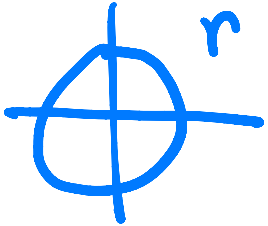
b)  $x=a \rightsquigarrow z=a+iy$ y varies

$a=0, \pm 1, \pm 2$

d) $(re^{i\phi})^2 = r^2 e^{2i\phi}$

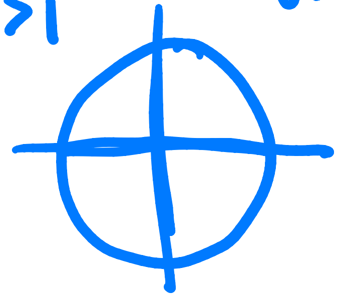
$|z^2| = r^2 = |z|^2$

$\arg(z^2) = 2\phi = 2\arg(z)$

e) 

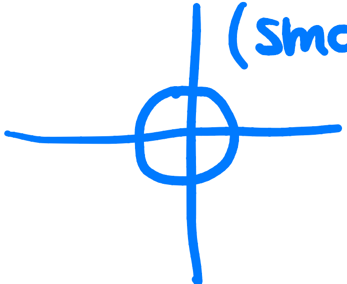
r

$r>1$ (bigger)

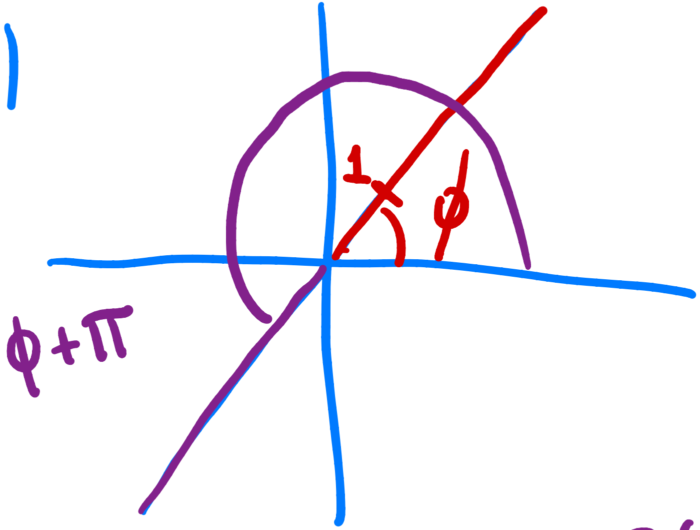


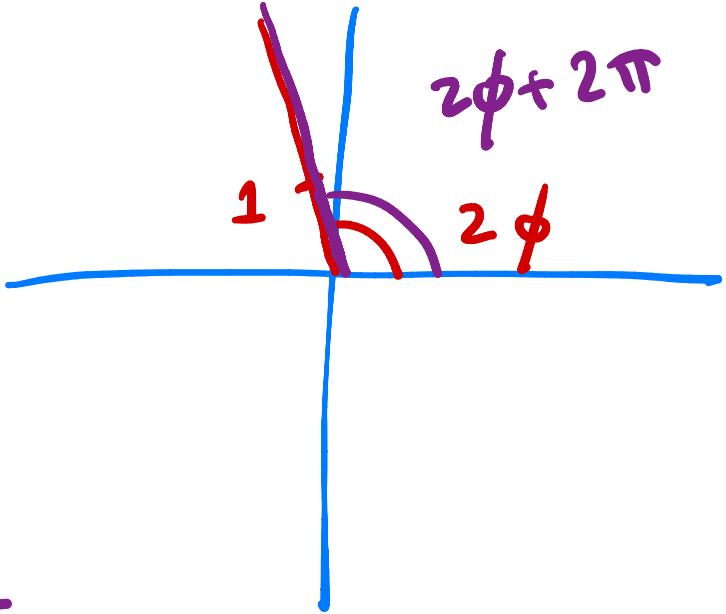
r^2

$r<1$ (smaller)



f)





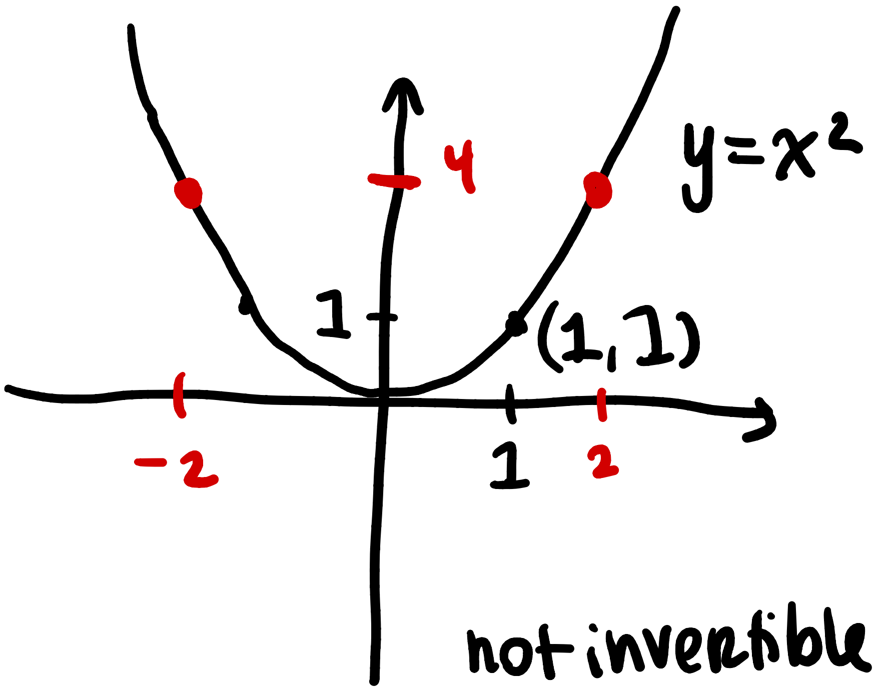
$180^\circ = \pi$ radians

$2(\phi+\pi)$
$= 2\phi + 2\pi$

g) h) i)

Branches of functions

Real number $f(x) = x^2$



not invertible

not injective

We cannot define a function $g: \mathbb{R} \to \mathbb{R}$ that is the inverse of $f: \mathbb{R} \to \mathbb{R}$

because

$g(4) = 2$ and -2

Recall that for g to be an inverse for f

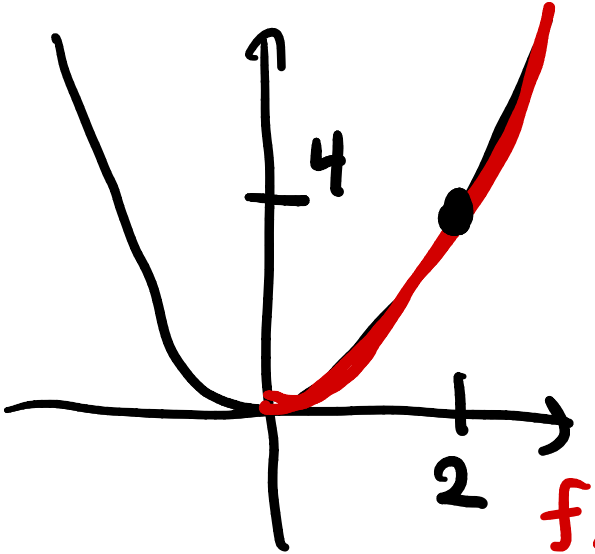
we would need $g(f(2)) = 2$

and $g(f(-2)) = -2$

but $f(2) = f(-2) = 4$ so need $g(4) = 2$

$g(4) = -2$

The solution we have chosen is to fix a branch of the square root.



$f(x) = x^2$

$f : [0, \infty) \to \mathbb{R}$

$f(x) = x^2$

injective

$g(x) = \sqrt{x}$

is the inverse of this

f (with domain $[0, \infty)$)

$\sqrt{4} = 2$

We have all agreed / we were all told that

$\sqrt{x}$ means "the" <u>positive</u> square root,

leads to issues like $\sqrt{x^2} = \sqrt{4}$

$$x = 2$$

Alternative world, we may have chosen the negative square root to be "the" square root.

$$\sqrt{4} = -2$$

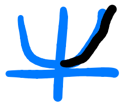
Alternatively, we may have not chosen at all, we might have defined the symbol $\sqrt{x}$ to be "a" square root of x; either number s.t. when squared we get x. Then $\sqrt{x}$ would be multivalued. We would write

$$\sqrt{4} = \pm 2$$

this is false in our world $\sqrt{4} = 2$

When faced with a function that is not injective, but for which we want an "inverse"

$f(x)=x^2$



2 choices

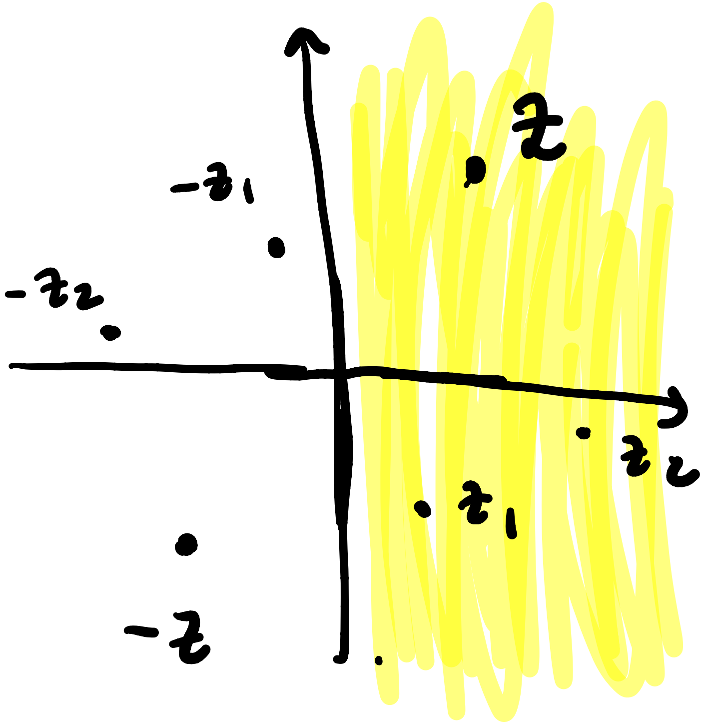
① Pick one inverse and call it "the" inverse ($\sqrt{4}=2$)

⇝ choosing a principal branch (for "the" square root)

② Call all of the inverses "an" inverse ($\sqrt{4}=\pm 2$)

⇝ keep all branches

What to do with $f(z) = z^2$?



looking for w with

$$w^2 = z^2$$

$$w = -z$$

If we restrict f to have domain

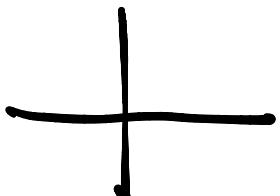
$$U = \{ z \in \mathbb{C} : \arg(z) \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \}$$

then f is injective

The principal branch of $\sqrt{z}$ is the inverse to this function.

Alternatively, we can say that $\sqrt{z}$ is the unique complex number w with

$$w^2 = z \quad \text{and} \quad \arg(w) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

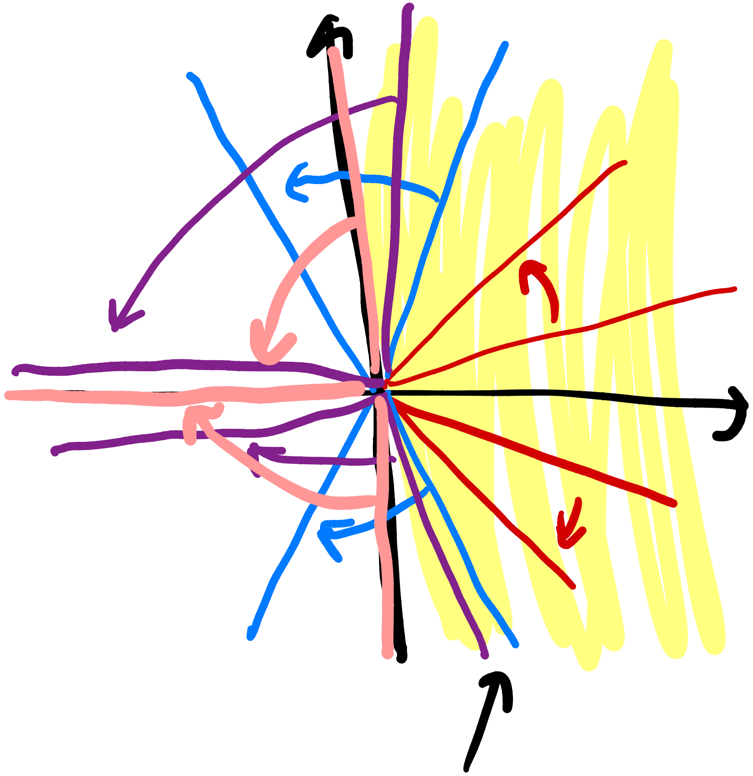


Also domain $V = \{ z \in \mathbb{C} : \arg(z) \in (-\frac{\pi}{2}, \frac{\pi}{2}] \}$

makes f injective

$\sqrt{-1} = i$ not $-i$

↖ principal branch



$-\frac{\pi}{2} < \arg(z) < \frac{\pi}{2}$

Under $f(z) = z^2$

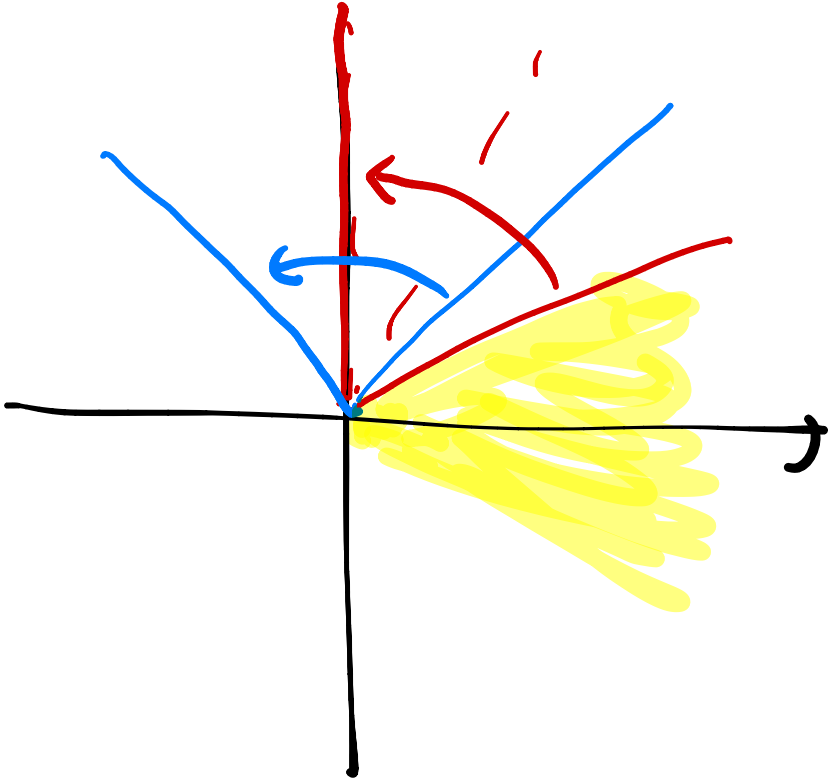
the right half plane

in yellow stretches

to fill all of $\mathbb{C}$ − neg real

axis

$f(z) = z^3$



THAT'S ALL FOR TODAY!

OH: 12pm-1pm

Campuswire