Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

Check in

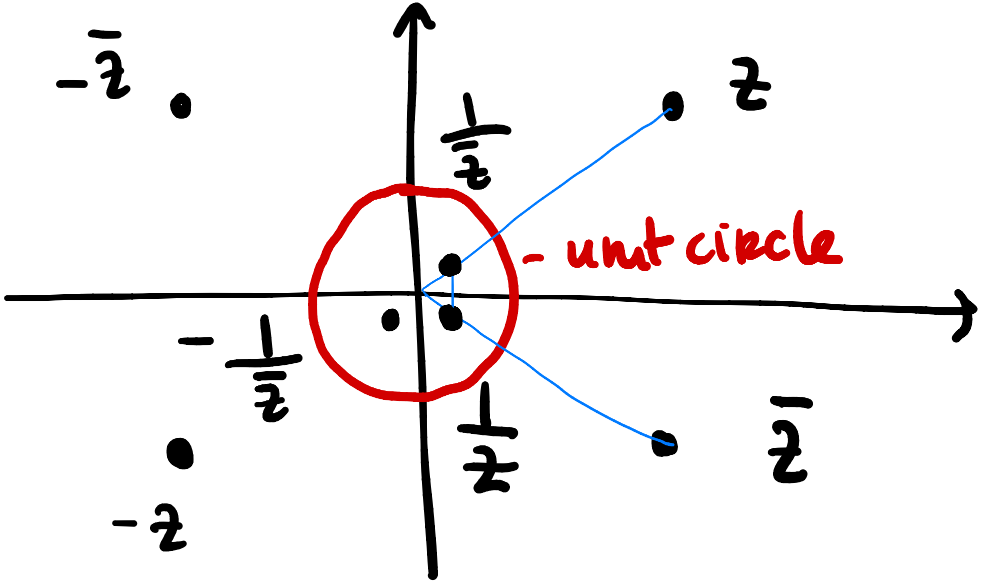
Any questions or concerns? Anything unclear?

HW 4 #1, 3, 4

#1, #3 → at the end of today

#4 → Friday 1st thing 4b)

#4a)



$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

$$\frac{1}{\bar{z}} = \overline{\left(\frac{1}{z}\right)}$$

$$-\frac{1}{\bar{z}} = -\overline{\left(\frac{1}{z}\right)}$$

Recap on fractional linear transformations

$f(z) = \frac{az+b}{cz+d}$

$ad - bc \neq 0$

$f(z) = \frac{az+b}{d} = \cancel{\frac{a}{d}}_{a} \cdot z + \cancel{\frac{b}{d}}\, b$

- If $c = 0$, then $f(z) = az + b$, and $f: \mathbb{C} \to \mathbb{C}$ is a bijection.
- If $c \neq 0$, then $f: \mathbb{C} - \{\frac{-d}{c}\} \to \mathbb{C} - \{\frac{a}{c}\}$ is a bijection.

More facts on fractional linear transformations

Warm up 4.1 $f(z) = z + i$ translation

#2 $f(z) = (1+i)z$ rotation + scaling ...

- If $c = 0$, f is a composition of translations, rotations, and scalings. $= az + b$

- If $c \neq 0$, f is a composition of translations, rotations, scalings, and inversions.

$f(z) = \dfrac{az+b}{cz+d}$

(This is BMPS Proposition 3.3.)

MORE FACTS ON FRACTIONAL LINEAR TRANSFORMATIONS

BMPS Theorem 3.4

- If $c = 0$, f sends lines to lines and circles to circles.
- If $c \neq 0$, f sends (lines and circles) to (lines and circles.)

to show this, enough to show $f(z) = \frac{1}{z}$ does this

lines → lines
lines → circles
circles → circles (but not center → center)
circles → lines

Notice

- If $c = 0$, f is nice. Sometimes we call it an affine function.
- If $c \neq 0$, everything is super annoying.

HOW TO FIX YOUR PROBLEMS MATH-STYLE

$$f(z)=\frac{az+b}{cz+d}$$

$c \neq 0$

$$\mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}$$

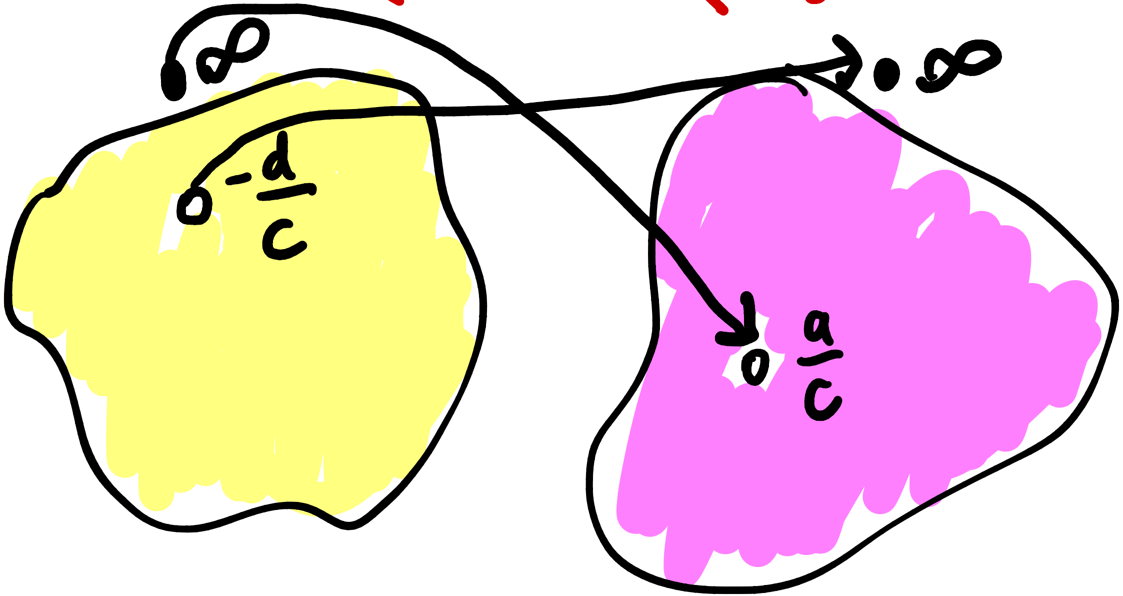
$$f : \cancel{\mathbb{C}-\left\{\frac{-d}{c}\right\}} \to \cancel{\mathbb{C}-\left\{\frac{a}{c}\right\}}$$

bijection

say

$f(\infty)=\frac{a}{c}$

$f\left(\frac{-d}{c}\right)=\infty$



THIS IS NOT SO SILLY!

say $f(z) = \dfrac{az+b}{cz+d}$ $\quad c \neq 0$

$f(\infty) = \dfrac{a}{c}$

$f\left(-\dfrac{d}{c}\right) = \infty$

- $\displaystyle\lim_{z \to \infty} \frac{az+b}{cz+d} = \lim_{z \to \infty} \frac{a + \cancel{\frac{b}{z}}^{\,0}}{c + \cancel{\frac{d}{z}}^{\,0}} = \frac{a}{c}$

- $\displaystyle\lim_{z \to -\frac{d}{c}} \frac{az+b}{cz+d} = \infty$

What if $c=0$?? $f(z)=az+b$ $\lim_{z\to\infty}(az+b)=\infty$

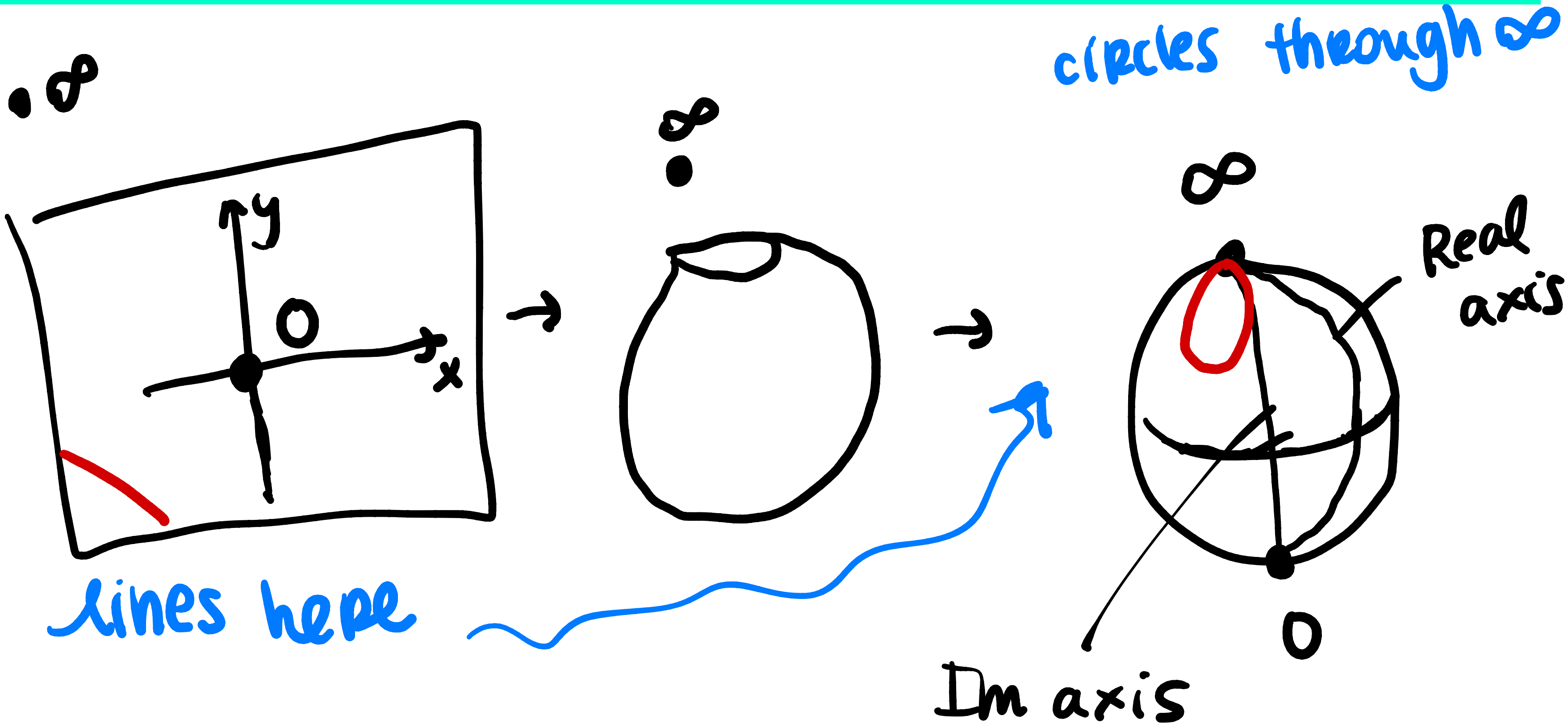
$$f(\infty)=\infty$$

We can say that $f(z)=\dfrac{az+b}{cz+d}$ $(c=0 \text{ or } c\neq 0)$

is a bijection $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

THE COMPLEX SPHERE OR EXTENDED COMPLEX PLANE

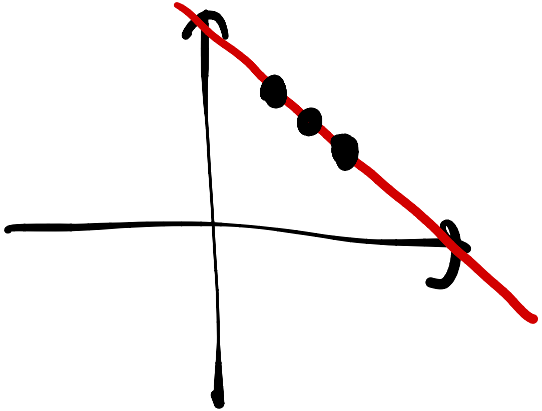


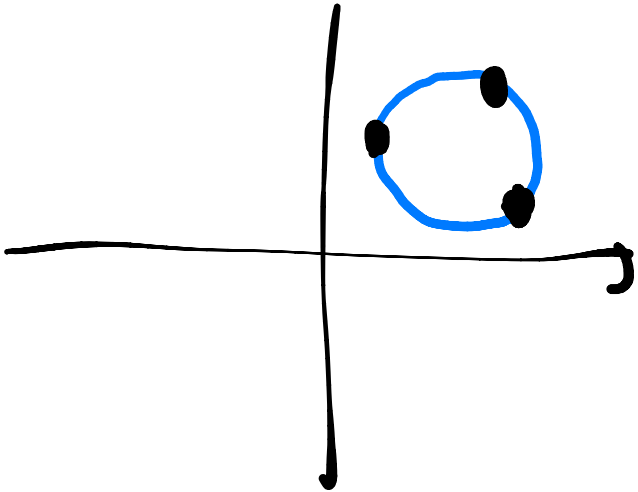
MORE FACTS ON THE COMPLEX SPHERE or Riemann sphere

- On the sphere, the lines are circles! So fractional linear transformations are bijections of the extended complex plane that send circles to circles.
- Fractional linear transformations are exactly the bijective conformal maps of the sphere to itself.
- The affine transformations are the ones that fix infinity.
- All fractional linear transformations can be obtained by rotating and moving the sphere and then doing the projection again.

Important tip!

In the plane, three points determine either a line or a circle; on the sphere three points determine a circle.





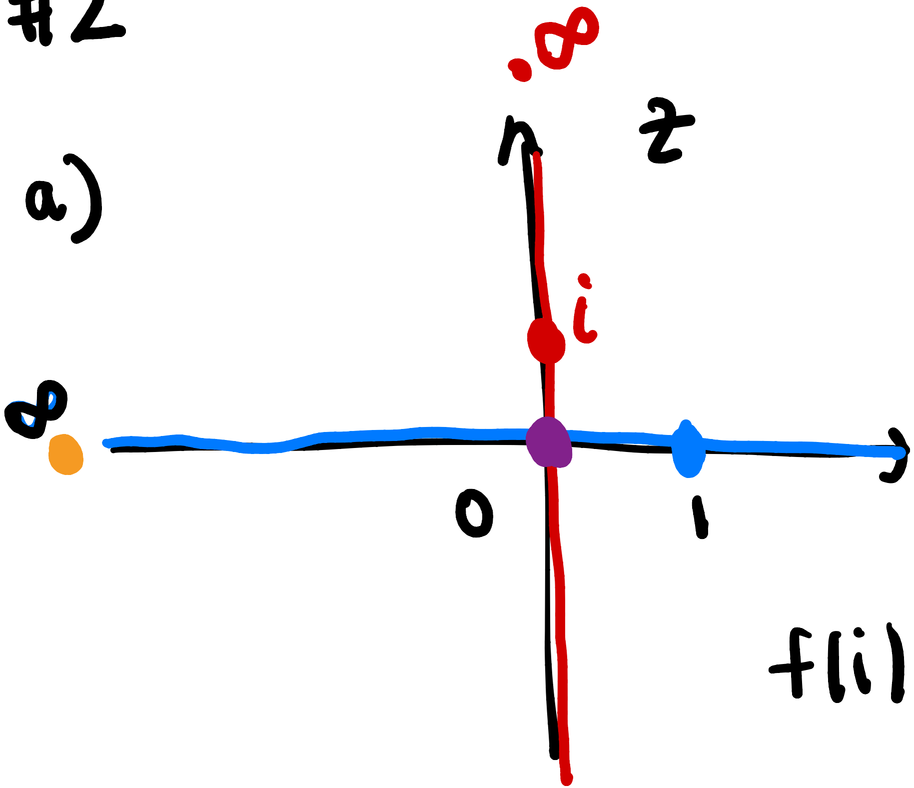
Back to HW #2

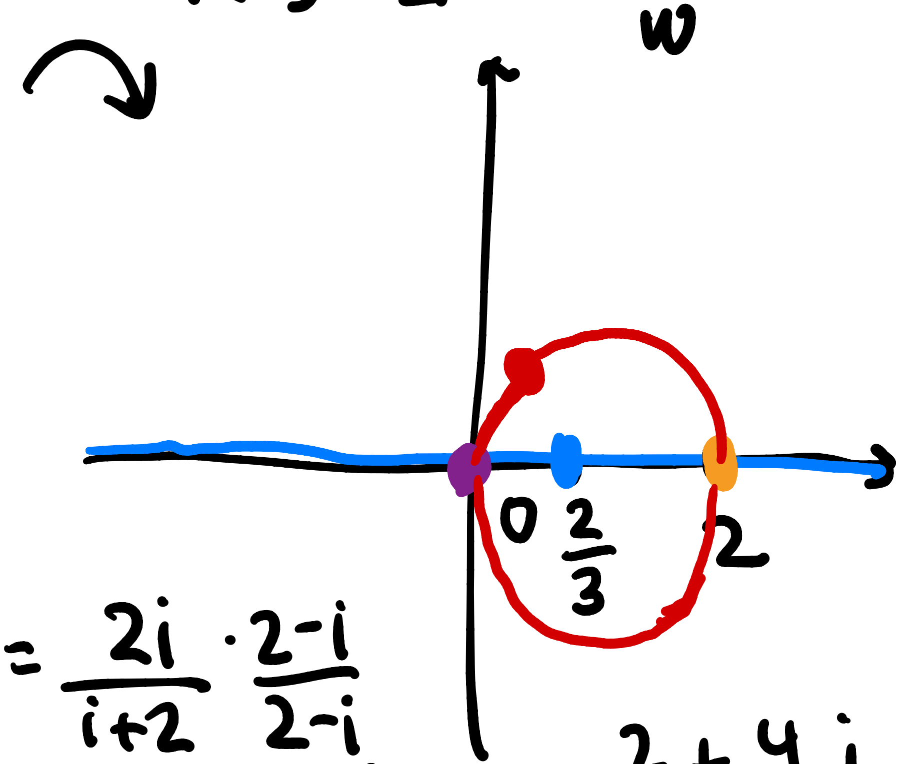
$w = \dfrac{2z}{z+2}$

$f(0) = \dfrac{0}{2} = 0 \quad f(1) = \dfrac{2}{3}$

$f(\infty) = 2$

a)





$f(i) = \dfrac{2i}{i+2} \cdot \dfrac{2-i}{2-i}$

$= \dfrac{4i+2}{1+4} = \dfrac{2}{5} + \dfrac{4}{5} i$

Friday: work the HW

if you need can turn in HW
until Sunday

THAT'S ALL FOR TODAY!

OR ask on Campuswire!

OH today 3-5pm