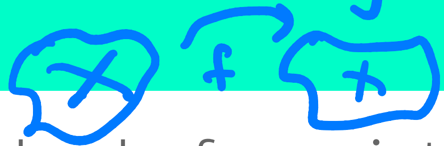
Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

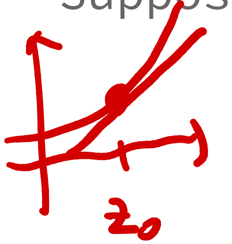
HOLOMORPHIC IS BASICALLY CONFORMAL

preserves angle



$f'(z_0) \neq 0$

Suppose that f is holomorphic in a neighborhood of a point.

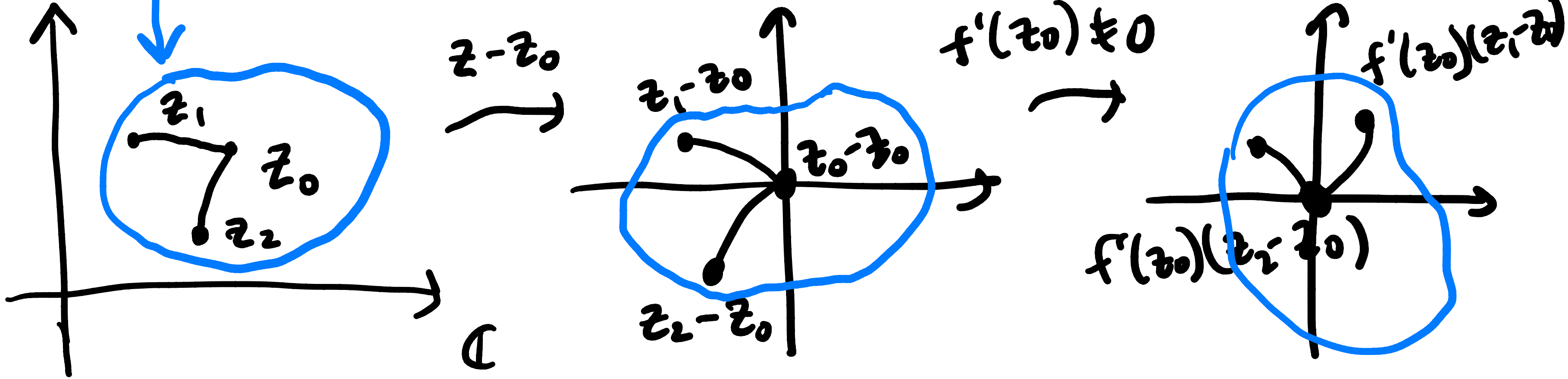


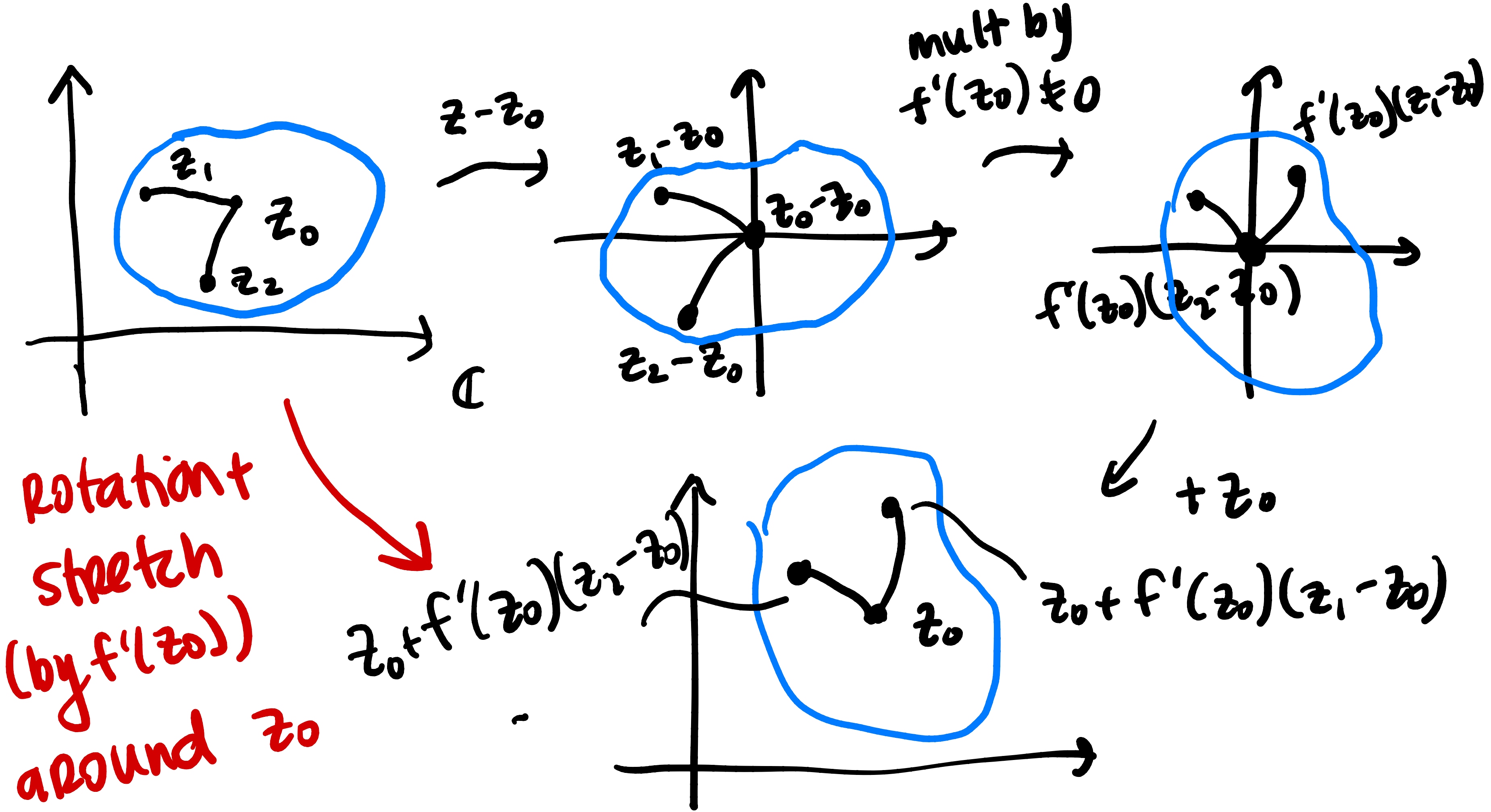
equation "holds"

$$f(z) \approx z_0 + f'(z_0)(z - z_0)$$

line tangent to f at z_0

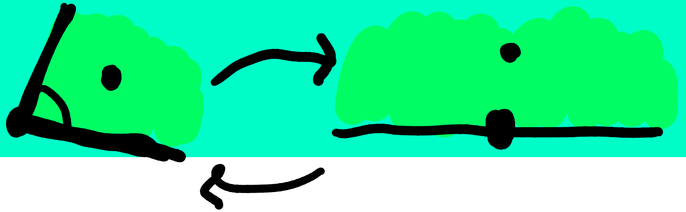
if z is near z_0 then



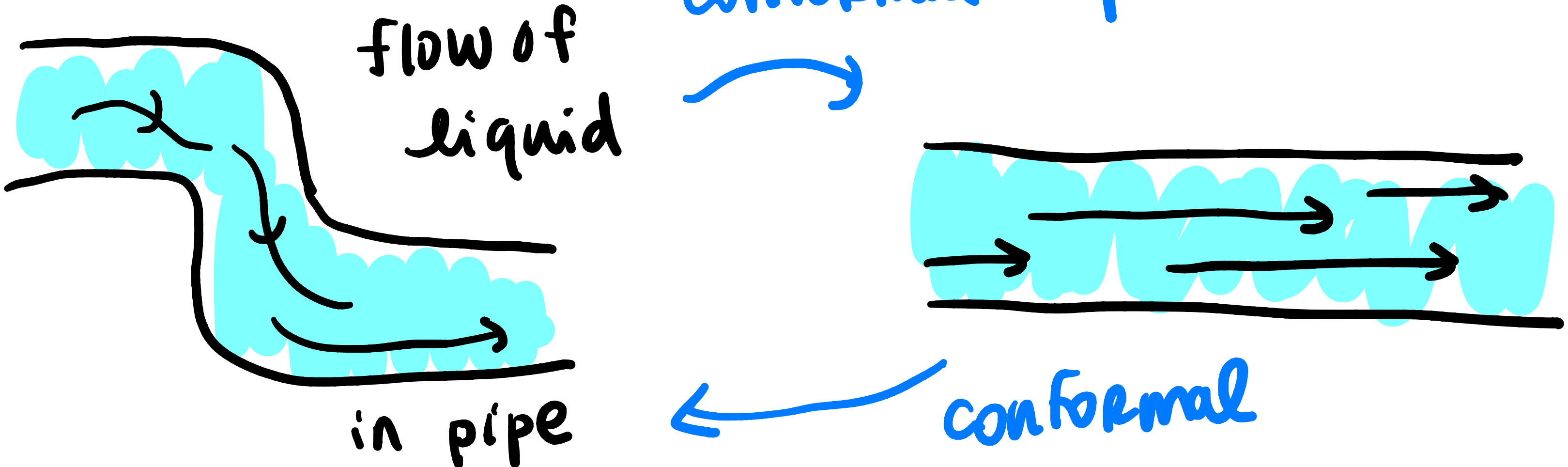


z_1
z_0
z_2
$\mathbb{C}$
$z - z_0$
$z_1 - z_0$
$z_0 - z_0$
$z_2 - z_0$
mult by $f'(z_0) \neq 0$
$f'(z_0)(z_1 - z_0)$
$f'(z_0)(z_2 - z_0)$
$+ z_0$
$z_0 + f'(z_0)(z_2 - z_0)$
z_0
$z_0 + f'(z_0)(z_1 - z_0)$
Rotation + stretch (by $f'(z_0)$) around z_0

CONFORMAL MAPS ARE COOL!



- Conformal maps were once a hot area of mathematical study, and are still studied to this day.
- They have helped solve problems in physics by moving situations with annoying geometries to situations with good geometries!



HOLOMORPHIC FUNCTIONS AND HARMONIC FUNCTIONS

If f = u + i v is holomorphic, then u and v are harmonic functions.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

this is true

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Theorem

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$ if cts

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

HARMONIC FUNCTIONS $u(x,y): \mathbb{R}^2 \to \mathbb{R}$

Harmonic functions are **real** functions, like in calculus or real analysis. Usually real functions are not so nice. But harmonic functions are the real functions that are nice!

They are still a hot area of mathematical research now. In addition, they give solutions to many classical physics problems (electrical and gravitational potential problems for example).

Converse to the Cauchy-Riemann equations

C-R

f is diff at z_0 $\Rightarrow$ $\boxed{\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \text{ exist}}$ $\boxed{\dfrac{\partial f}{\partial x} = -i\dfrac{\partial f}{\partial y}}$

Note that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ might not be continuous!

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f is diff at z_0 $\Leftarrow$ $\boxed{\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \text{ exist}}$ and $\boxed{\text{are continuous}}$ around z_0

$$\boxed{\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$$

C-R

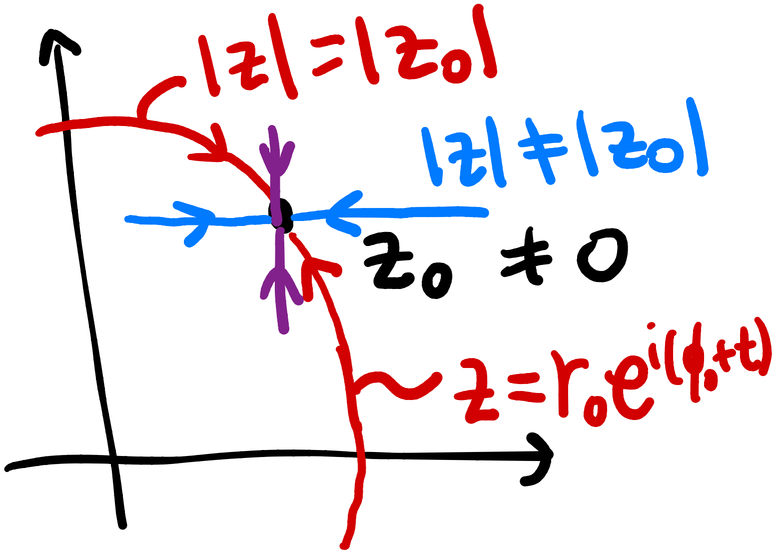
Check in

Any questions or concerns? Anything unclear?

HW 3 #1 $f(z) = |z|^2$

a) use limit def to show differentiability

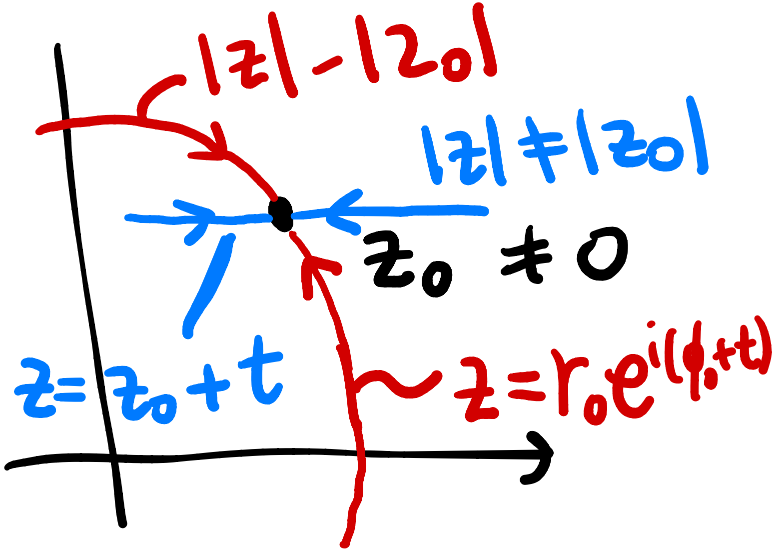
d) use the converse of C-R to show differentiability



$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{|z|^2-|z_0|^2}{z-z_0}$$

$$\lim_{z\to z_0}\frac{|z|^2-|z_0|^2}{z-z_0}=\lim_{z\to z_0}\frac{|r_0e^{i(\phi_0+t)}|^2-|r_0e^{i\phi_0}|^2}{z-z_0}$$

$$=\lim_{z\to z_0}\frac{r_0^2-r_0^2}{z-z_0}=\lim_{z\to z_0}0=0$$



$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{|z|^2-|z_0|^2}{z-z_0}$$

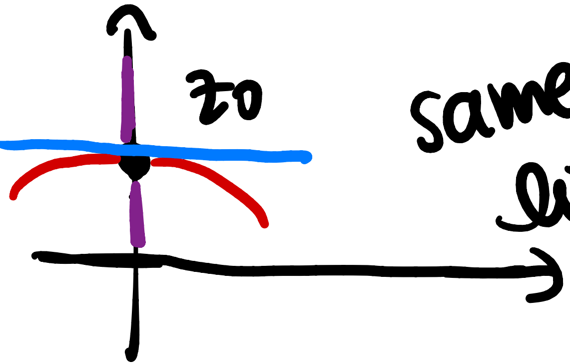
$z_0+t=(x_0+t)+iy_0$

$$\lim_{z\to z_0}\frac{|z|^2-|z_0|^2}{z-z_0}=\lim_{z\to z_0}\frac{|z_0+t|^2-|z_0|^2}{(z_0+t)-z_0}$$

$$=\lim_{z\to z_0}\frac{[(x_0+t)^2+y_0^2]-[x_0^2+y_0^2]}{t}=\lim_{z\to z_0}\frac{2x_0t+t^2}{t}$$

$$\lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0} = \lim_{t\to 0} \frac{2x_0t+t^2}{t}$$

$$= \lim_{t\to 0} (2x_0+t) = 2x_0$$



same limit!

if $x_0 \neq 0$ then f is not diff

If $x_0=0$ $(\mathrm{Re}(z_0)=0)$ then these 2 give same answer so need the 3rd path z_0+it

THAT'S ALL FOR TODAY!

See you on Campuswire!