Math 295 - Fall 2020
Homework 3
Due at 11:59pm on Friday September 25

Please turn in this assignment on Gradescope.

Problem 1 : (Objectives B1,B2,B3)

a) Use the limit definition of derivative to show that the function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = |z|^2$ is only differentiable at $z_0 = 0$. (Hint: For $z_0 \neq 0$, approach $z_0 = r_0 e^{i\phi_0}$ first along the path $z(t) = r_0 e^{i(\phi_0 + t)}$, $t \to 0$, and then along the paths $z(t) = z_0 + t$ and $z(t) = z_0 + it$, $t \to 0$.)

b) Is f holomorphic? If so, on what region(s) is f holomorphic?

c) Does f satisfy the Cauchy-Riemann equations?

d) Use the converse of the Cauchy-Riemann equations to prove that f is only differentiable at $z_0 = 0$.

Problem 2 : (Objective B4) Let $U \subseteq \mathbb{C}$ be an open set, and $f\colon U \to \mathbb{C}$ be holomorphic. Write $f(z) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ take real values, as usual. Show that the gradient vectors

$$\nabla u(x,y) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad \text{and} \quad \nabla v(x,y) = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

have dot product zero with each other at every point of U, and that $\nabla u(x, y) = 0$ if and only if $\nabla v(x, y) = 0$.

Problem 3 : (Objective B5) Prove that if $f(z)$ and $\overline{f(z)}$ are both holomorphic in the region $U \subseteq \mathbb{C}$, then $f(z)$ is constant on U.

Problem 4 : (Objective B4,B5) Suppose that f is entire and can be written as $f(z) = u(x) + iv(y)$, that is, the real part of f depends only on $x = \mathrm{Re}(z)$ and the imaginary part of f depends only on $y = \mathrm{Im}(z)$. Prove that $f(z) = az + b$ for some $a \in \mathbb{R}$ and some $b \in \mathbb{C}$.