- all my office hours this week are canceled
- this week there is an objective for graduate

Complex Analysis

This lecture will be recorded. If you do not want your face in the recording, please turn off your camera. If you do not want your voice in the recording, please participate using the chat.

RECENT HISTORY

f is nice enough

Theorem

Let f be a complex function. If f is holomorphic on an annulus centered at z_0, then f has a Laurent series expansion of the form

$$f(z) = \sum_{k \in \mathbb{Z}} c_k (z - z_0)^k,$$

where

$$c_k = \frac{1}{2\pi i} \int_\gamma f(z) \frac{1}{(z - z_0)^{k+1}} \, dz.$$

"we say integrate f against $(z-z_0)^{-(k+1)}$"

Pros and Cons

Pros:

- Easy to integrate and differentiate
- Easy to see poles and zeros, nice region of convergence

Cons:

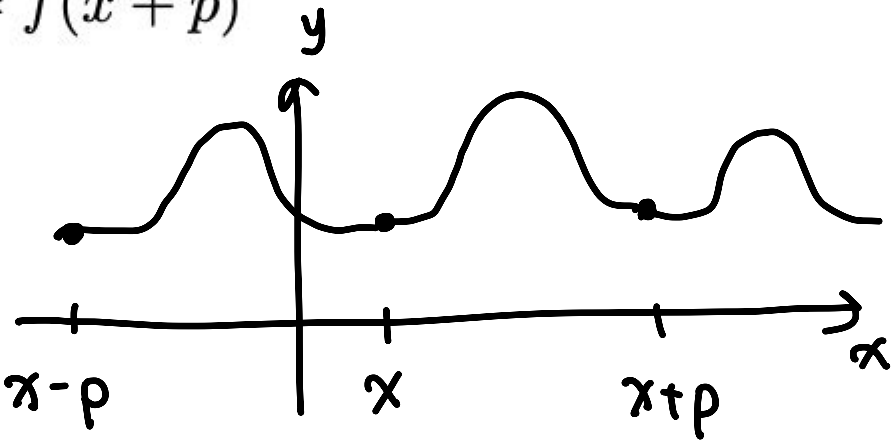
- Doesn’t converge very quickly for an important class of functions: *periodic* functions

Periodic Functions

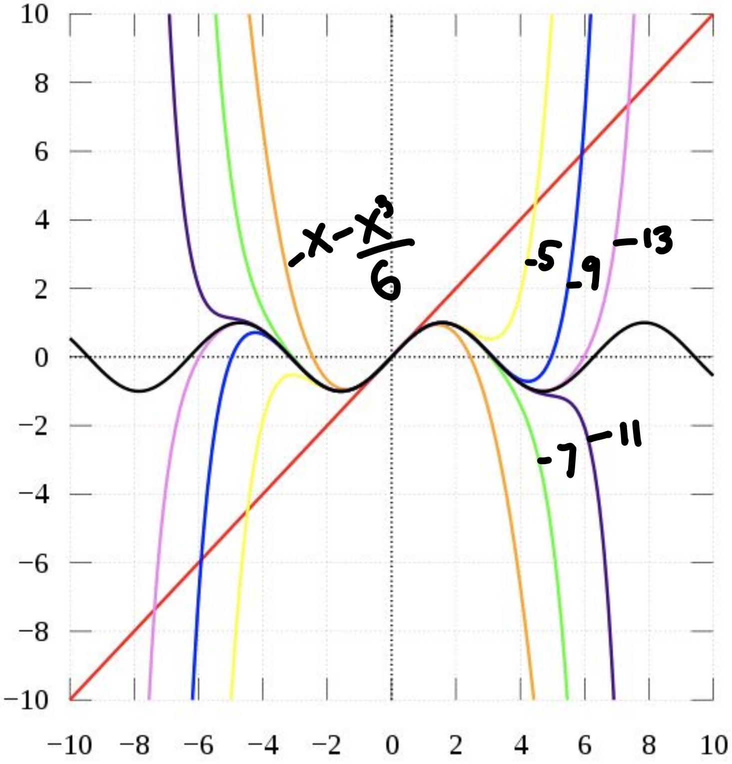
A real function f is periodic if there is a number p, called the period, such that

$$f(x) = f(x+p)$$

for all x in the domain of f.



Example



$p = 2\pi$

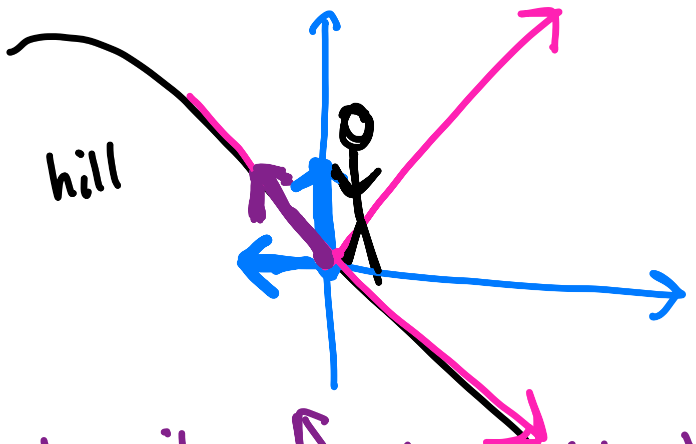
This is the function sin(x), with its Taylor polynomials of degrees 1, 3, 5, 7, 9, 11, and 13.

This image is due to IkamusumeFan and I found it on Wikipedia.

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

HOW TO DO BETTER

Think of the elements ... $x^{-2}, x^{-1}, 1, x, x^2$ ... as a *basis* in which we are expressing the function.



to describe in the pink "frame of reference" requires only one number because one of the pink directions agrees with the direction of motion

describe in the blue "frame of reference" need 2 numbers: how much up and how much to the left

HOW TO DO BETTER e^{ikx} has period $\frac{2\pi}{|k|}$

Maybe we can come up with a better basis? For example, for periodic functions, we could choose a basis that is periodic!

For simplicity, assume the period is 2π (this is no big deal since we can stretch and shrink in the x-direction)

THE periodic function: e^{ix} $\{e^{ikx} : k \in \mathbb{Z}\}$

$\ldots, e^{-2ix}, e^{-ix}, 1, e^{ix}, e^{2ix}, \ldots$

Dirichlet's Conditions

$\int_{-\pi}^{\pi} |f| \, dx$

Theorem

Let f be a real periodic function of period 2π, and suppose that f is absolutely integrable over a period, has bounded variation in any bounded interval, has only finite discontinuities, and has a finite number of discontinuities in any bounded interval.

f is nice enough

Then f has a Fourier series

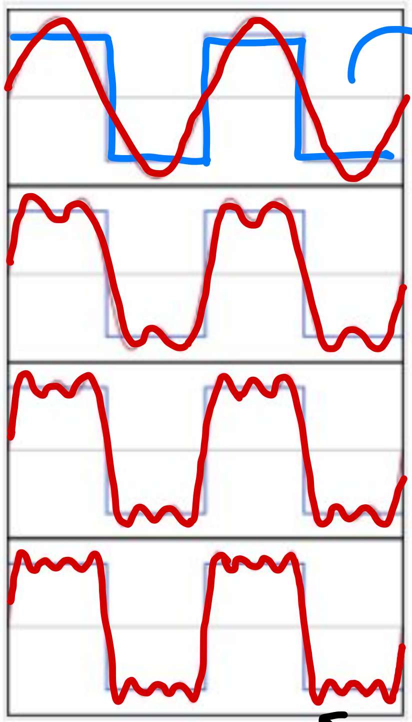
$$f(x) = \sum_{k \in \mathbb{Z}} a_k e^{ikx},$$

where

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \frac{1}{e^{ikx}} \, dx.$$

Better convergence

$a_{-1}e^{-ix} + a_0 + a_1 e^{ix}$



$$f(x) = \ldots + a_{-4}e^{-4ix} + a_{-3}e^{-3ix} + a_{-2}e^{-2ix} + a_{-1}e^{-ix} + a_0 + a_1 e^{ix} + a_2 e^{2ix} + a_3 e^{3ix} + a_4 e^{4ix} + \ldots$$

These are the first four partial sums of the Fourier series for a square wave.

This image is due to Jim.Belk and I found it on Wikipedia.

Gibb's phenomenon

Aside on transforms

$$\int_{-}^{-} f(x) \, \square \, dx$$

This idea of integrating f “against” some function is extremely fruitful and has given rise to a lot of interesting math.

You can look up “list of Fourier-related transforms” on Wikipedia to read more.

1st topic

Fourier transforms and differential equations

In solving differential equations we can often use the *Fourier transform*, which is defined to be

$\frac{d^2}{dx} f(x) \ldots$

$k^2 F(k)$

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

$f''(x) + 2f'(x) + f(x) = e^x$

$(k^2 + 2k + 1) F(k) = \widehat{e^x}$

This is useful because it "changes" differentiation into multiplication, so certain linear equations become just polynomials to solve.

$F(k) = \frac{\widehat{e^x}}{k^2 + 2k + 1}$

Efficient representation of waves

The Fourier series gives an efficient representation of waves: In many phenomenon only certain coefficients are nonzero (or significant) so by “throwing out” the small coefficients we can represent a wave without too much information.

Image and sound compression

2nd topic - least technical

MP3s

This has allowed the development of "lossy" compression algorithms that give us small music and image files.

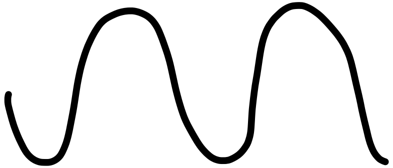
3rd topic

WAVELETS

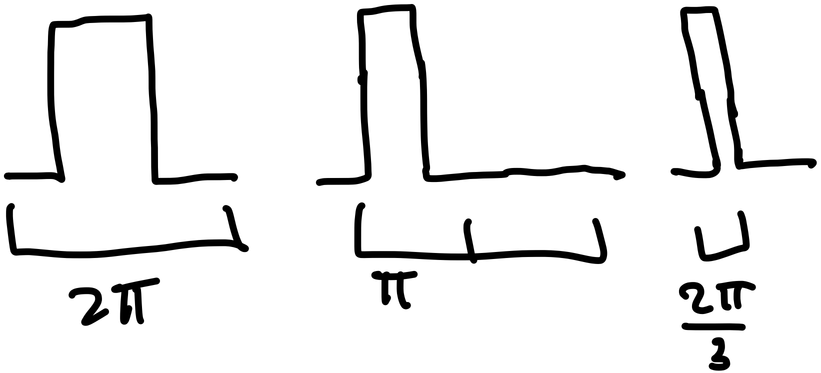
There are in fact even better bases than the exponential functions. For example, in 1986 Daubechies developed her wavelets which are both first-order accurate and orthogonal.

accuracy

efficient



$$e^{ix} = \cos x + i \sin x$$



THAT'S ALL FOR TODAY!

Grad credit

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