Three.I Isomorphisms

Linear Algebra
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Example We can think of $\mathcal{M}_{2\times 2}$ as "the same" as $\mathbb{R}^4$ if we associate in this way.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

For instance, these are corresponding elements.

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix}$$

This association persists under addition.

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

$$\longleftrightarrow \quad \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \\ d_1 + d_2 \end{pmatrix}$$

Isomorphism

1.3 *Definition* An *isomorphism* between two vector spaces V and W is a map $f\colon V \to W$ that

1) is a correspondence: f is one-to-one and onto;
2) *preserves structure:* if $\vec{v}_1, \vec{v}_2 \in V$ then

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$$

and if $\vec{v} \in V$ and $r \in \mathbb{R}$ then

$$f(r\vec{v}) = rf(\vec{v})$$

(we write $V \cong W$, read "V is isomorphic to W", when such a map exists).

How-to

To verify that a function $f\colon V \to W$ between two vector spaces is an isomorphism, do these four.

- To show that f is one-to-one, assume that $\vec{v}_1, \vec{v}_2 \in V$ are such that $f(\vec{v}_1) = f(\vec{v}_2)$ and derive that $\vec{v}_1 = \vec{v}_2$.
- To show that f is onto, assume that $\vec{w} \in W$ and find a $\vec{v} \in V$ such that $f(\vec{v}) = \vec{w}$.
- To show that f preserves addition, check that for all $\vec{v}_1, \vec{v}_2 \in V$ we have $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$.
- To show that f preserves scalar multiplication, check that for all $\vec{v} \in V$ and $r \in \mathbb{R}$ we have $f(r \cdot \vec{v}) = r \cdot f(\vec{v})$.

The intuition behind the first two is to ensure that the spaces correspond: for each member of W there exactly one assocated member of V. For the latter two, we've seen some initial discussion above, and the next section develops the ideas at length.

Example The space of quadratic polynomials $\mathcal{P}_2$ is isomorphic to $\mathbb{R}^3$ under this map.

$$f(a_0 + a_1 x + a_2 x^2) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

Here are two examples of the action of f.

$$f(1 + 2x + 3x^2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad f(3 + 4x^2) = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

To verify that f is an isomorphism we must check condition (1), that f is a correspondence, and condition (2), that f preserves structure.

The first part of (1) is that f is one-to-one. We usually verify one-to-one-ness by supposing that the function yields the same output on two inputs $f(a_0 + a_1x + a_2x^2) = f(b_0 + b_1x + b_2x^2)$ and from that derive that the two inputs must be equal. The definition of f gives

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

and two column vectors are equal only if their entries are equal $a_0 = b_0$, $a_1 = b_1$, and $a_2 = b_2$. Thus the original inputs are equal $a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$. So f is one-to-one.

The second part of (1) is that f is onto. We usually verify onto-ness by considering an element of the codomain

$$\vec{w} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^3$$

and producing an element of the domain that maps to it. Observe that $\vec{w}$ is the image under f of the member $\vec{v} = a_0 + a_1x + a_2x^2$ of the domain. Thus f is onto.

Condition (2) also has two halves. First we must show that f preserves addition. Consider f acting on the sum of two elements of the domain.

$$f(\,(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)\,)$$
$$= f(\,(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2\,)$$

The definition of f gives

$$= \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

and that equals

$$= \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

which gives

$$= f(\,a_0 + a_1x + a_2x^2) + f(b_0 + b_1x + b_2x^2\,)$$

as required.

We finish by checking that f preserves scalar multiplication. This is similar to the check for addition.

$$\begin{aligned} r \cdot f(\, a_0 + a_1 x + a_2 x^2 \,) &= r \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} ra_0 \\ ra_1 \\ ra_2 \end{pmatrix} \\ &= f(\, (ra_0) + (ra_1)x + (ra_2)x^2 \,) \end{aligned}$$

So the function f is an isomorphism. Because there is an isomorphism, the two spaces are isomorphic $\mathcal{P}_2 \cong \mathbb{R}^3$.

1.10 *Lemma* An isomorphism maps a zero vector to a zero vector.

Proof Where $f\colon V \to W$ is an isomorphism, fix some $\vec{v} \in V$. Then $f(\vec{0}_V) = f(0 \cdot \vec{v}) = 0 \cdot f(\vec{v}) = \vec{0}_W$. QED

1.11 *Lemma* For any map $f\colon V \to W$ between vector spaces these statements are equivalent.

(1) f preserves structure

$$f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2) \quad \text{and} \quad f(c\vec{v}) = c\,f(\vec{v})$$

(2) f preserves linear combinations of two vectors

$$f(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2)$$

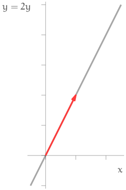
(3) f preserves linear combinations of any finite number of vectors

$$f(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

The book contains the proof's details.

This result eases checking that a function preserves the structure of a vector space, since we can do it in one step with statement (2).

Example This line through the origin



$$L = \{ \begin{pmatrix} t \\ 2t \end{pmatrix} = t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid t \in \mathbb{R} \}$$

is a vector space under the addition and scalar multiplication operations that it inherits from $\mathbb{R}^2$.

$$\begin{pmatrix} t_1 \\ 2t_1 \end{pmatrix} + \begin{pmatrix} t_2 \\ 2t_2 \end{pmatrix} = \begin{pmatrix} t_1 + t_2 \\ 2(t_1 + t_2) \end{pmatrix} \qquad r \cdot \begin{pmatrix} t \\ 2t \end{pmatrix} = \begin{pmatrix} rt \\ 2rt \end{pmatrix}$$

We will verify that

$$f(\begin{pmatrix} t \\ 2t \end{pmatrix}) = t$$

is an isomorphism between L and $\mathbb{R}^1$.

To verify that f is one-to-one suppose that f maps two members of L to the same output.

$$f(\,t_1\cdot\begin{pmatrix}1\\2\end{pmatrix}\,)=f(\,t_2\cdot\begin{pmatrix}1\\2\end{pmatrix}\,)$$

By the definition of f we have that $t_1 = t_2$ and so the two members of L are equal.

To check that f is onto consider a member of the codomain, $r \in \mathbb{R}$. There is a member of the domain L that maps to it, namely this one.

$$f(\,r\cdot\begin{pmatrix}1\\2\end{pmatrix}\,)$$

To finish, check that f preserves structure with the lemma's (2).

$$\begin{aligned}f(\,c_1\cdot\begin{pmatrix}t_1\\2t_1\end{pmatrix}+c_2\cdot\begin{pmatrix}t_2\\2t_2\end{pmatrix}\,)&=f(\,(c_1t_1++c_2t_2)\cdot\begin{pmatrix}1\\2\end{pmatrix}\,)\\&=c_1t_1+c_2t_2=c_1\cdot f(\,\begin{pmatrix}t_1\\2t_1\end{pmatrix}\,)+c_2\cdot f(\,\begin{pmatrix}t_2\\2t_2\end{pmatrix}\,)\end{aligned}$$

Dimension characterizes isomorphism

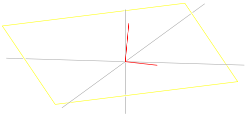
2.3 *Theorem* Vector spaces are isomorphic if and only if they have the same dimension.

The proof is these two lemmas.

2.4 *Lemma* If spaces are isomorphic then they have the same dimension.

2.5 *Lemma* If spaces have the same dimension then they are isomorphic.

Example The plane $2x - y + z = 0$ through the origin in $\mathbb{R}^3$ is a vector space (under the natural addition and scalar multiplication operations).



Consider that to be a one-equation linear system and parametrize $x = (1/2)y - (1/2)z$ to describe the space as a span.

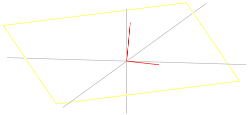
$$P = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} z \mid y, z \in \mathbb{R} \}$$

Clearly that two-vector set is linearly independent, so it is a basis.

$$B = \langle \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \rangle$$

The basis B has two vectors so this is a dimension 2 space. For instance, it is isomorphic to $\mathbb{R}^2$.

Example Consider again the plane



$$P = \{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot y + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot z \mid y, z \in \mathbb{R} \}$$

The second lemma's proof shows that this is an isomorphism: the map $f\colon P \to \mathbb{R}^2$ that associates each element $\vec{v} \in P$ with its representation $\mathrm{Rep}_B(\vec{v}) \in \mathbb{R}^2$. Here is an example of its action.

$$\vec{v}_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 3 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot (-4) \quad \stackrel{f}{\longmapsto} \quad \mathrm{Rep}_B(\vec{v}_2) = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Another example.

$$\vec{v}_2 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot (1/2) + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 9 \quad \stackrel{f}{\longmapsto} \quad \mathrm{Rep}_B(\vec{v}_2) = \begin{pmatrix} 1/2 \\ 9 \end{pmatrix}$$

The first lemma's proof shows that where we take the domain to have basis vectors $\vec{\beta}_i$ then under an isomorphism f the images $f(\vec{\beta}_i)$ form a basis for the range. Applied here we have

$$\vec{\beta}_1 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 1 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 0 \quad \stackrel{f}{\longmapsto} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\vec{\beta}_2 = \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} \cdot 0 + \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \cdot 1 \quad \stackrel{f}{\longmapsto} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which together make the basis $\mathcal{E}_2$ for $\mathbb{R}^2$.