Reduces to is an equivalence

1.5 *Lemma* Elementary row operations are reversible.

Proof For any matrix A, the effect of swapping rows is reversed by swapping them back, multiplying a row by a nonzero k is undone by multiplying by $1/k$, and adding a multiple of row i to row j (with $i \neq j$) is undone by subtracting the same multiple of row i from row j.

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} \xrightarrow{\rho_j \leftrightarrow \rho_i} A \qquad A \xrightarrow{k\rho_i} \xrightarrow{(1/k)\rho_i} A \qquad A \xrightarrow{k\rho_i + \rho_j} \xrightarrow{-k\rho_i + \rho_j} A$$

(The third case requires that $i \neq j$.) QED

We say that matrices that reduce to each other are equivalent with respect to the relationship of row reducibility. The next result justifies this, using the definition of an equivalence.

1.6 *Lemma* Between matrices, 'reduces to' is an equivalence relation.

The book has the full proof. For the basic idea, consider this Gauss's Method application.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -5 \end{pmatrix} \xrightarrow{-2\rho_1+\rho_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & -3 \end{pmatrix}$$

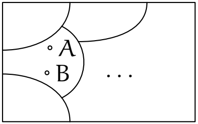
While our initial experience applying Gauss's Method leads us to feel that the second matrix in some way "comes after" the first, in fact the two are inter-reducible. Here are some other 2×3 matrices that are inter-reducible with those two.

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 & -2 \\ 2 & 4 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & -1 \\ 3 & 6 & -6 \end{pmatrix}$$

In general, the collection of all matrices breaks into classes of inter-reducible matrices.

1.7 *Definition* Two matrices that are interreducible by elementary row operations are *row equivalent*.

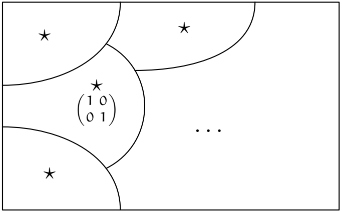
The diagram below shows the collection of all matrices as a box. Inside that box each matrix lies in a class. Matrices are in the same class if and only if they are interreducible. The classes are disjoint — no matrix is in two distinct classes. We have partitioned the collection of matrices into *row equivalence classes*.



2.6 *Theorem* Each matrix is row equivalent to a unique reduced echelon form matrix.

The book contains the full proof.

So the reduced echelon form is a canonical form for row equivalence: the reduced echelon form matrices are representatives of the classes.



Example To decide if these two are row equivalent

$$\begin{pmatrix}3 & 2 & 0\\1 & -1 & 2\\4 & 1 & 2\end{pmatrix} \qquad \begin{pmatrix}3 & 1 & -2\\6 & 2 & -4\\1 & 0 & 2\end{pmatrix}$$

use Gauss-Jordan elimination to bring each to reduced echelon form and see if they are equal. The results are

$$\xrightarrow[-(4/3)\rho_1+\rho_3]{-(1/3)\rho_1+\rho_2} \quad \xrightarrow{-1\rho_2+\rho_3} \quad \xrightarrow[-(3/5)\rho_2]{(1/3)\rho_1} \quad \xrightarrow{-(2/3)\rho_2+\rho_1} \begin{pmatrix}1 & 0 & 4/5\\0 & 1 & -6/5\\0 & 0 & 0\end{pmatrix}$$

and

$$\xrightarrow[-(1/3)\rho_1+\rho_3]{-2\rho_1+\rho_2} \quad \xrightarrow{\rho_2\leftrightarrow\rho_3} \quad \xrightarrow[-3\rho_2]{(1/3)\rho_1} \quad \xrightarrow{-(1/3)\rho_2+\rho_1} \begin{pmatrix}1 & 0 & 2\\0 & 1 & -8\\0 & 0 & 0\end{pmatrix}$$

and therefore the original matrices are not row equivalent.