Computing row rank / column rank
our approach vs the book

Our strategy:

To compute a basis for the column space

- get A into echelon form
- each column with a free variable corresponds to a superfluous column, throw it out
- basis is guaranteed if we keep columns corresponding to leading variables

To compute a basis for the row space, do this for A^T (A transpose)

→ Since this method shrinks the spanning set to a basis, the basis is a subset of the vectors we had.

The book strategy:

- The book technique computes a basis for the <u>row space</u>.
- The basis we get is <u>not</u> a subset of the original set. It's just some other vectors (that's ok).

The book technique relies on 2 theorems

1. Row operations do not change the row space
2. The rows of a matrix in echelon form are linearly independent.

To compute a basis for the <u>row space</u>

- get A in echelon form
- the non zero rows are a basis for the row space.

(For column space, do this for A^T)

Example: $B = \begin{pmatrix} 2 & 3 & 4 & 1 \\ -1 & 0 & -2 & 0 \\ 3 & 1 & 6 & 3 \\ 1 & 1 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 1 & -1 & 2 & 2 \end{pmatrix}$

Get B in echelon form:

$$\overset{\rho_1 \leftrightarrow \rho_4}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & -2 & 0 \\ 3 & 1 & 6 & 3 \\ 2 & 3 & 4 & 1 \\ 0 & 4 & 0 & 0 \\ 1 & -1 & 2 & 2 \end{pmatrix} \underset{\substack{\rho_4 - 2\rho_1 \\ \rho_6 - \rho_1}}{\overset{\substack{\rho_2 + \rho_1 \\ \rho_3 - 3\rho_1}}{\sim}} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix}$$

$$\underset{\substack{\rho_4 - \rho_2 \\ \rho_5 - 4\rho_2 \\ \rho_6 + 2\rho_2}}{\overset{\rho_3 + 2\rho_2}{\sim}} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \overset{\rho_3 \leftrightarrow \rho_4}{\sim} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\begin{array}{l} \rho_4 - 3\rho_3 \\ \sim \\ \rho_6 - 2\rho_3 \end{array} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is echelon form

(but not <u>reduced</u> echelon form; we do need to go that far)

The non zero rows are

$$(1\ 1\ 2\ 0)^T$$
$$(0\ 1\ 0\ 0)^T$$
$$(0\ 0\ 0\ 1)^T$$

So these form a basis for the row space of B. The rank of B is 3.

Compare with our method:

$$B = A^T, \qquad A = \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

The row space of B is the column space of A. We computed a basis for the column space on November 2

The basis was $\left\{ \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 6 \\ 3 \end{pmatrix} \right\}$

This is not the same basis as $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

But that is ok, a space has several bases.

However, both bases must have the same number of elements because the rank does not care about the method used to compute it.

Upshot: The first step of both methods is getting echelon form, so we can quickly give a basis for row space & column space.

Example: Let $A = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 3 & 8 \\ 0 & 1 & 2 \\ 4 & 0 & 4 \end{pmatrix}$

Give a basis for the column space of A and a basis for the row space of A.

Step 1: Echelon form:

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 3 & 8 \\ 0 & 1 & 2 \\ 4 & 0 & 4 \end{pmatrix} \underset{\rho_4 - 4\rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \begin{pmatrix} 1 & 3 & 7 \\ 0 & -3 & -6 \\ 0 & 1 & 2 \\ 0 & -12 & -24 \end{pmatrix} \overset{\rho_2 \leftrightarrow \rho_3}{\sim} \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \\ 0 & -12 & -24 \end{pmatrix}$$

$$\underset{\rho_4 + 12\rho_2}{\overset{\rho_3 + 3\rho_2}{\sim}} \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is echelon form.

Column space: variable in 3rd column is free so a basis for column space is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$

Row space: Take non zero rows:

$$\left\{ (1\ 3\ 7)^T,\ (0\ 1\ 2)^T \right\}$$

is a basis for row space

For both the column & the row space, our justification to argue we did find a basis is that the process we used guarantees it.