Name:

Problem 1: *Find a set that spans the subspace*

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : 2x + y + w = 0 \text{ and } y + 2z = 0 \right\}$$

of $\mathbb{R}^4$.

Hint: Start with solving the system of two equations $2x + y + w = 0$ *and* $y + 2z = 0$.

Solution: Recall that the solutions to a homogeneous system of linear equations have the form

$$c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_k\vec{\beta}_k,$$

where the c_i's range over all real numbers and the $\vec{\beta}_i$ are some vectors. This is exactly what it means for the vectors $\vec{\beta}_1 \ldots \vec{\beta}_k$ to *span* the set of solutions of the system. So we will just do as usual and use the vectors we get.
We start by solving the system. Because this is a homogeneous system, we omit the column of zeroes.

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \overset{\rho_1 - \rho_2}{\sim} \begin{pmatrix} 2 & 0 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \overset{\frac{1}{2}\rho_1}{\sim} \begin{pmatrix} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

We see that z and w are free, and $x = z - \frac{1}{2}w$ and $y = -2z$. In vector form this is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{pmatrix} w$$

From this, we can see that the set

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

spans the subspace of solutions to these two linear equations.