Example This is a basis for $\mathbb{R}^3$.

$$\mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Calculus books sometimes call those $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$.

1.5 *Definition* For any $\mathbb{R}^n$

$$\mathcal{E}_n = \langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \rangle$$

is the *standard* (or *natural*) basis. We denote these vectors $\vec{e}_1, \dots, \vec{e}_n$.

Checking that $\mathcal{E}_n$ is a basis for $\mathbb{R}^n$ is routine.

Although a basis is a sequence we will follow the common practice and refer to it as a set.

1.12 *Theorem* In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

Proof A sequence is a basis if and only if its vectors form a set that spans and that is linearly independent. A subset is a spanning set if and only if each vector in the space is a linear combination of elements of that subset in at least one way. Thus we need only show that a spanning subset is linearly independent if and only if every vector in the space is a linear combination of elements from the subset in at most one way.

Example Above we saw that in $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$. As part of that we computed the coefficients needed to express a member of $\mathcal{P}_1$ as a combination of basis vectors.

$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2,\ c_2 = (a - b)/2$$

For instance, the polynomial $3 + 4x$ has this expression

$$3 + 4x = (7/2) \cdot (1 + x) + (-1/2) \cdot (1 - x)$$

so its representation is this.

$$\mathrm{Rep}_B(3 + 4x) = \begin{pmatrix} 7/2 \\ -1/2 \end{pmatrix}$$