1.3 *Corollary* For $\vec{v} \in S$, omitting that vector does not shrink the span $[S] = [S - \{\vec{v}\}]$ if and only if it is dependent on other vectors in the set $\vec{v} \in [S]$.

Example These two subsets of $\mathbb{R}^3$ have the same span

$$\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \} \qquad \{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \}$$

because in the first set $\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$.

1.13 *Corollary* A set S is linearly independent if and only if for any $\vec{v} \in S$, its removal shrinks the span $[S - \{v\}] \subsetneq [S]$.

Proof This follows from Corollary 1.3 . If S is linearly independent then none of its vectors is dependent on the other elements, so removal of any vector will shrink the span. If S is not linearly independent then it contains a vector that is dependent on other elements of the set, and removal of that vector will not shrink the span. QED

1.14 *Lemma* Suppose that S is linearly independent and that $\vec{v} \notin S$. Then the set $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin [S]$.

Proof We will show that $S \cup \{\vec{v}\}$ is not linearly independent if and only if $\vec{v} \in [S]$.

Suppose first that $v \in [S]$. Express $\vec{v}$ as a combination $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$. Rewrite that $\vec{0} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n - 1 \cdot \vec{v}$. Since $v \notin S$, it does not equal any of the $\vec{s}_i$ so this is a nontrivial linear dependence among the elements of $S \cup \{\vec{v}\}$. Thus that set is not linearly independent.

Now suppose that $S \cup \{\vec{v}\}$ is not linearly independent and consider a nontrivial dependence among its members $\vec{0} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n + c_{n+1} \cdot \vec{v}$. If $c_{n+1} = 0$ then that is a dependence among the elements of S, but we are assuming that S is independent, so $c_{n+1} \neq 0$. Rewrite the equation as $\vec{v} = (c_1/c_{n+1})\vec{s}_1 + \cdots + (c_n/c_{n+1})\vec{s}_n$ to get $\vec{v} \in [S]$

QED

1.16 *Corollary* In a vector space, any finite set has a linearly independent subset with the same span.

Proof If $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependent, S contains a vector $\vec{v}_1$ that is a linear combination of the others. Define the set $S_1 = S - \{\vec{v}_1\}$. By Corollary 1.3 the span does not shrink $[S_1] = [S]$.

If S_1 is linearly independent then we are done. Otherwise iterate: take a vector $\vec{v}_2$ that is a linear combination of other members of S_1 and discard it to derive $S_2 = S_1 - \{\vec{v}_2\}$ such that $[S_2] = [S_1]$. Repeat this until a linearly independent set S_j appears; one must appear eventually because S is finite and the empty set is linearly independent. QED

Example Consider this subset of $\mathbb{R}^2$.

$$S = \{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4, \vec{s}_5\} = \{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \}$$

The linear relationship

$$r_1 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + r_2 \begin{pmatrix} 3 \\ 3 \end{pmatrix} + r_3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + r_4 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + r_5 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives a system of equations.

$$\begin{array}{rcl} 2r_1 + 3r_2 + r_3 \phantom{{}-r_4} + r_5 &=& 0 \\ 2r_1 + 3r_2 + 4r_3 - r_4 - r_5 &=& 0 \end{array}$$

$$\xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} 2r_1 + 3r_2 + r_3 \phantom{{}-r_4} + r_5 &=& 0 \\ +3r_3 - r_4 - 2r_5 &=& 0 \end{array}$$

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\{\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R}\}$$

Set $r_5 = 1$ and $r_2 = r_4 = 0$ to get $r_1 = -5/6$ and $r_3 = 2/3$,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that $\vec{s}_5$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Similarly, setting $r_4 = 1$ and the other parameters to 0 shows $\vec{s}_4$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Also, setting $r_2 = 1$ and the other parameters to 0 shows $\vec{s}_2$ is in the span of the same set. So we can omit the vectors $\vec{s}_2$, $\vec{s}_4$, $\vec{s}_5$ associated with the free variables without shrinking the span. The set $\{\vec{s}_1, \vec{s}_3\}$ is linearly independent and so we cannot omit any members without shrinking the span.