3.14 *Corollary* Where the matrix A is $n \times n$, these statements

(1) the rank of A is n
(2) A is nonsingular
(3) the rows of A form a linearly independent set
(4) the columns of A form a linearly independent set
(5) any linear system whose matrix of coefficients is A has one and only one solution

are equivalent.

Proof Clearly (1) $\iff$ (2) $\iff$ (3) $\iff$ (4). The last, (4) $\iff$ (5), holds because a set of n column vectors is linearly independent if and only if it is a basis for $\mathbb{R}^n$, but the system

$$c_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

has a unique solution for all choices of $d_1, \ldots, d_n \in \mathbb{R}$ if and only if the vectors of a's on the left form a basis. QED

Row space

3.1 *Definition* The *row space* of a matrix is the span of the set of its rows. The *row rank* is the dimension of this space, the number of linearly independent rows.

3.3 *Lemma* If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

Proof Corollary One.III.2.4 shows that when $A \longrightarrow B$ then each row of B is a linear combination of the rows of A. That is, in the above terminology, each row of B is an element of the row space of A. Then $\text{Rowspace}(B) \subseteq \text{Rowspace}(A)$ follows because a member of the set $\text{Rowspace}(B)$ is a linear combination of the rows of B, so it is a combination of combinations of the rows of A, and by the Linear Combination Lemma is also a member of $\text{Rowspace}(A)$.

For the other set containment, recall Lemma One.III.1.5 , that row operations are reversible so $A \longrightarrow B$ if and only if $B \longrightarrow A$. Then $\mathrm{Rowspace}(A) \subseteq \mathrm{Rowspace}(B)$ follows as in the previous paragraph.

QED

3.3 *Lemma* The nonzero rows of an echelon form matrix make up a linearly independent set.

Proof Lemma One.III.2.5 says that no nonzero row of an echelon form matrix is a linear combination of the other rows. This result just restates that in this chapter's terminology. QED

Example The matrix before Gauss's Method and the matrix after have equal row spaces.

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ -1 & -2 & 2 & 2 & 0 \\ 2 & 4 & 5 & 2 & 9 \end{pmatrix} \xrightarrow[-2\rho_1+\rho_3]{\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows of the latter matrix form a basis for Rowspace(M).

$$B = \langle \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3 & 2 & 3 \end{pmatrix} \rangle$$

The row rank is 2.

So Gauss's Method produces a basis for the row space of a matrix. It has found the "repeat" information, that M's third row is three times the first plus the second, and eliminated that extra row.

Transpose

3.8 *Definition* The *transpose* of a matrix is the result of interchanging its rows and columns, so that column j of the matrix A is row j of A^{T} and vice versa.

Example To find a basis for the column space of a matrix,

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

transpose,

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix}$$

reduce,

$$\begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix} \xrightarrow{(-3/2)\rho_1+\rho_2} \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

and transpose back.

$$\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}^{\mathsf{T}} = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$$

This basis

$$B = \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

shows that the column space is the entire vector space $\mathbb{R}^2$.