Three.IV Matrix Operations

Linear Algebra

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Sums and Scalar Products

Definition of matrix sum and scalar multiple

1.3 *Definition* The *scalar multiple* of a matrix is the result of entry-by-entry scalar multiplication. The *sum* of two same-sized matrices is their entry-by-entry sum.

Example Where

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 2 \\ 9 & -1/2 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 8 & -1 \end{pmatrix}$$

Then

$$A + C = \begin{pmatrix} 2 & -1 \\ 10 & 2 \end{pmatrix} \qquad 5B = \begin{pmatrix} 0 & 0 & 10 \\ 45 & -5/2 & 25 \end{pmatrix}$$

None of these is defined: $A + B$, $B + A$, $B + C$, $C + B$, because the sizes don't match.

Matrix Multiplication

Representing composition

Another function operation, besides scalar multiplication and addition, is composition.

2.1 *Lemma* The composition of linear maps is linear.

Proof Let $h\colon V \to W$ and $g\colon W \to U$ be linear. The calculation

$$\begin{aligned} g\circ h\,\bigl(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2\bigr) &= g\bigl(\,h(c_1\cdot\vec{v}_1+c_2\cdot\vec{v}_2)\,\bigr) = g\bigl(\,c_1\cdot h(\vec{v}_1)+c_2\cdot h(\vec{v}_2)\,\bigr) \\ &= c_1\cdot g\bigl(h(\vec{v}_1)\bigr)+c_2\cdot g\bigl(h(\vec{v}_2)\bigr) = c_1\cdot(g\circ h)(\vec{v}_1)+c_2\cdot(g\circ h)(\vec{v}_2) \end{aligned}$$

shows that $g\circ h\colon V\to U$ preserves linear combinations, and so is linear. QED

We next do an exploratory calculation to see how the matrix representations of the two functions combine to make the matrix representation of their composition.

Example Consider two linear functions $h\colon V\to W$ and $g\colon W\to X$ represented as here.

$$\mathrm{Rep}_{B,C}(h)=\begin{pmatrix}3&1\\2&5\\4&6\end{pmatrix}\qquad \mathrm{Rep}_{C,D}(g)=\begin{pmatrix}8&7&11\\9&10&12\end{pmatrix}$$

We want to see how these two representations combine to give the representation of the map $g\circ h\colon V\to X$.

We start with the action of h on a $\vec{v}\in V$.

$$\begin{aligned}\mathrm{Rep}_{C}(h(\vec{v}))&=\mathrm{Rep}_{B,C}(h)\cdot\mathrm{Rep}_{B}(\vec{v})\\&=\begin{pmatrix}3&1\\2&5\\4&6\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}3v_1+v_2\\2v_1+5v_2\\4v_1+6v_2\end{pmatrix}\end{aligned}$$

Of course we represent application of h by doing the matrix vector multiplication.

Next apply g.

$$\mathrm{Rep}_{C,D}(g)\cdot\mathrm{Rep}_{C}(h(\vec{v})) = \begin{pmatrix} 8 & 7 & 11 \\ 9 & 10 & 12 \end{pmatrix}\begin{pmatrix} 3v_1+v_2 \\ 2v_1+5v_2 \\ 4v_1+6v_2 \end{pmatrix}$$

$$= \begin{pmatrix} 8(3v_1+v_2)+7(2v_1+5v_2)+11(4v_1+6v_2) \\ 9(3v_1+v_2)+10(2v_1+5v_2)+12(4v_1+6v_2) \end{pmatrix}$$

Gather terms.

$$= \begin{pmatrix} (8\cdot3+7\cdot2+11\cdot4)v_1+(8\cdot1+7\cdot5+11\cdot6)v_2 \\ (9\cdot3+10\cdot2+12\cdot4)v_1+(9\cdot1+10\cdot5+12\cdot6)v_2 \end{pmatrix}$$

Rewrite as a matrix-vector multiplication.

$$= \begin{pmatrix} 8\cdot3+7\cdot2+11\cdot4 & 8\cdot1+7\cdot5+11\cdot6 \\ 9\cdot3+10\cdot2+12\cdot4 & 9\cdot1+10\cdot5+12\cdot6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

So here is how the two starting matrices combine.

$$\begin{pmatrix} 8 & 7 & 11 \\ 9 & 10 & 12 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 8\cdot3+7\cdot2+11\cdot4 & 8\cdot1+7\cdot5+11\cdot6 \\ 9\cdot3+10\cdot2+12\cdot4 & 9\cdot1+10\cdot5+12\cdot6 \end{pmatrix}$$

Definition of matrix multiplication

2.3 *Definition* The *matrix-multiplicative product* of the $m \times r$ matrix G and the $r \times n$ matrix H is the $m \times n$ matrix P, where

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

so that the i,j-th entry of the product is the dot product of the i-th row of the first matrix with the j-th column of the second.

$$GH = \begin{pmatrix} & \vdots & & \\ g_{i,1} & g_{i,2} & \cdots & g_{i,r} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} & h_{1,j} & \\ \cdots & h_{2,j} & \cdots \\ & \vdots & \\ & h_{r,j} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & p_{i,j} & \cdots \\ & \vdots & \end{pmatrix}$$

Example

$$\begin{pmatrix} 3 & 1 & 6 \\ 2 & 5 & 9 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 1 & -3 & 5 \\ 4 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 31 & 9 & 59 \\ 45 & 3 & 96 \end{pmatrix}$$

Example This product is not defined because the number of columns on the left must equal the number of rows on the right.

$$\begin{pmatrix}1 & 3 & -1\\ 0 & 0 & 0\\ 2 & 0 & 0\end{pmatrix}\begin{pmatrix}5 & 7 & 1\\ 2 & 2 & 0\end{pmatrix}$$

Example Square matrices of the same size have a defined product.

$$\begin{pmatrix}1 & 3 & -1\\ 0 & 0 & 0\\ 2 & 0 & 0\end{pmatrix}\begin{pmatrix}5 & 7 & 1\\ 2 & 2 & 0\\ 1 & -1 & 2\end{pmatrix}=\begin{pmatrix}10 & 14 & -1\\ 0 & 0 & 0\\ 10 & 14 & 2\end{pmatrix}$$

This reflects the fact that we can compose two functions from a space to itself $f, g\colon V \to V$.

Order, dimensions, and sizes

These two functions can be composed

$$h\colon V \to W \qquad g\colon W \to X$$

because the codomain of f is the domain of g.

Observe about the order of the notation: in writing the composition $g \circ h$, the function g is written first (meaning leftmost) but it is applied second.

$$\vec{v} \xmapsto{h} h(\vec{v}) \xmapsto{g} g(h(\vec{v}))$$

That order carries over to matrices: $g \circ h$ is represented by GH.

Also consider the dimensions of the spaces.

$$\text{dimension } n \text{ space} \xrightarrow{h} \text{dimension } r \text{ space} \xrightarrow{g} \text{dimension } m \text{ space}$$

The $m \times n$ matrix GH is the product of an $m \times r$ matrix G and a $r \times n$ matrix H. Briefly, $m \times r$ times $r \times n$ equals $m \times n$.

Matrix multiplication is not commutative

Function composition is in general not a commutative operation — $\cos(\sqrt{x})$ is different than $\sqrt{\cos(x)}$. This holds even in the special case of composition of linear functions.

Example Changing the order in which we multiply these matrices

$$\begin{pmatrix} 3 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 33 \\ 24 & 20 \end{pmatrix}$$

changes the result.

$$\begin{pmatrix} -2 & 6 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -6 & 18 \\ 18 & 38 \end{pmatrix}$$

Example The product of these matrices

$$\begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 8 & 12 & 0 \\ -4 & 0 & 1/2 \end{pmatrix}$$

is defined in one order and not defined in the other.

Although the matrix operation of multiplication does not have the property of being commutative, it does have some nice algebraic properties.

2.12 *Theorem* If F, G, and H are matrices, and the matrix products are defined, then the product is associative $(FG)H = F(GH)$ and distributes over matrix addition $F(G + H) = FG + FH$ and $(G + H)F = GF + HF$.

Proof Associativity holds because matrix multiplication represents function composition, which is associative: the maps $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as both send $\vec{v}$ to $f(g(h(\vec{v})))$.

Distributivity is similar. For instance, the first one goes $f \circ (g + h)\,(\vec{v}) = f\big(\,(g + h)(\vec{v})\,\big) = f\big(\,g(\vec{v}) + h(\vec{v})\,\big) = f(g(\vec{v})) + f(h(\vec{v})) = f \circ g(\vec{v}) + f \circ h(\vec{v})$ (the third equality uses the linearity of f). Right-distributivity goes the same way. QED