Math 124 - Fall 2016

Some linearly independent subsets to compute

For each of the following sets S, give a subset of S that spans the same set as S but that is linearly independent. The solutions start on the next page, so don't peek until you are done!

a) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix} \right\}$

b) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$

c) $S = \{-1 + 3x^2, -2 + 6x^2, 1 + x - 2x^2, -2 + x + 2x^2, 7 - 3x - 9x^2\}$

d) $S = \left\{ \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1/2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

Solutions

a) We first solve the equation

$$a_1 \begin{pmatrix}1\\2\\3\end{pmatrix} + a_2 \begin{pmatrix}4\\5\\6\end{pmatrix} + a_3 \begin{pmatrix}0\\1\\0\end{pmatrix} + a_4 \begin{pmatrix}1\\-3\\7\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

We do this by solving a homogenous system. Here we only give the row operations, and we omit the column of zeroes:

$$\begin{pmatrix}1 & 4 & 0 & 1\\2 & 5 & 1 & -3\\3 & 6 & 0 & 7\end{pmatrix} \underset{\rho_3-3\rho_1}{\overset{\rho_2-2\rho_1}{\sim}} \begin{pmatrix}1 & 4 & 0 & 1\\0 & -3 & 1 & -5\\0 & -6 & 0 & 4\end{pmatrix} \overset{\rho_3-2\rho_2}{\sim} \begin{pmatrix}1 & 4 & 0 & 1\\0 & -3 & 1 & -5\\0 & 0 & -2 & 14\end{pmatrix}$$

From this we see that a_1, a_2 and a_3 are leading and a_4 is free. We conclude that

$$\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-3\\7\end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

and

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

is linearly independent. In other words, the fourth vector was superfluous.

b) This time we start by solving the equation

$$a_1 \begin{pmatrix}1\\2\\0\end{pmatrix} + a_2 \begin{pmatrix}0\\1\\1\end{pmatrix} + a_3 \begin{pmatrix}1\\0\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Again we solve a homogeneous. We give the row operations and omit the column of zeroes:

$$\begin{pmatrix}1 & 0 & 1\\2 & 1 & 0\\0 & 1 & 3\end{pmatrix} \overset{\rho_2-2\rho_1}{\sim} \begin{pmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 1 & 3\end{pmatrix} \overset{\rho_3-\rho_2}{\sim} \begin{pmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 5\end{pmatrix}$$

All variables are leading. We conclude that the set was already linearly independent, and no vector can be thrown out.

c) This looks very different, but we will see that it turns out pretty much the same as before, after we do a few steps of simplifying the polynomials. We start by solving this equation:

$$a_1(-1+3x^2)+a_2(-2+6x^2)+a_3(1+x-2x^2)+a_4(-2+x+2x^2)+a_5(7-3x-9x^2) = 0.$$

We do this by collecting all like terms:

$$(-a_1 - 2a_2 + a_3 - 2a_4 + 7a_5) + (a_3 + a_4 - 3a_5)x + (3a_1 + 6a_2 - 2a_3 + 2a_4 - 9a_5)x^2 = 0.$$

Now for a polynomial to be zero, each term (the constant term, the linear term and the quadratic term) has to be zero individually. This gives us the following system of equations:

$$\begin{aligned} -a_1 - 2a_2 + a_3 - 2a_4 + 7a_5 &= 0 \\ a_3 + a_4 - 3a_5 &= 0 \\ 3a_1 + 6a_2 - 2a_3 + 2a_4 - 9a_5 &= 0 \end{aligned}$$

And so now we're in the familiar situation of solving a homogeneous system of linear equations. From here the problem will go the same as the others.We show the row operations and omit the column of zeroes, as always:

$$\begin{pmatrix} -1 & -2 & 1 & -2 & 7 \\ 0 & 0 & 1 & 1 & -3 \\ 3 & 6 & -2 & 2 & -9 \end{pmatrix} \underset{-\rho_1}{\overset{\rho_3+3\rho_1}{\sim}} \begin{pmatrix} 1 & 2 & -1 & 2 & -7 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -4 & 12 \end{pmatrix}$$
$$\overset{\rho_3-\rho_2}{\sim} \begin{pmatrix} 1 & 2 & -1 & 2 & -7 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & -5 & 15 \end{pmatrix}$$

Here a_2 and a_5 are free and a_1, a_3 and a_4 are leading. Therefore

$$\begin{aligned} &\text{Span}\{-1 + 3x^2, -2 + 6x^2, 1 + x - 2x^2, -2 + x + 2x^2, 7 - 3x - 9x^2\} \\ &\qquad = \text{Span}\{-1 + 3x^2, 1 + x - 2x^2, -2 + x + 2x^2\} \end{aligned}$$

and

$$\{-1 + 3x^2, 1 + x - 2x^2, -2 + x + 2x^2\}$$

is linearly independent.

d) This one also looks different, but again it'll be the same as the others once we get a homogeneous system of linear equations to solve. We start by solving this equation:

$$a_1 \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} + a_2 \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 & -1/2 \\ 0 & -1 \end{pmatrix} + a_4 \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} + a_5 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Addition in a matrix happens entry by entry. If we bring together the sum on the left, we get

$$\begin{aligned} &a_1 \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} + a_2 \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix} + a_3 \begin{pmatrix} 0 & -1/2 \\ 0 & -1 \end{pmatrix} + a_4 \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} + a_5 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 2a_1 & 2a_1 \\ 0 & 2a_1 \end{pmatrix} + \begin{pmatrix} -2a_2 & 0 \\ -a_2 & a_2 \end{pmatrix} + \begin{pmatrix} 0 & -1/2a_3 \\ 0 & -a_3 \end{pmatrix} + \begin{pmatrix} 4a_4 & a_4 \\ a_4 & -a_4 \end{pmatrix} + \begin{pmatrix} a_5 & a_5 \\ a_5 & a_5 \end{pmatrix} \\ &\quad = \begin{pmatrix} 2a_1 - 2a_2 + 4a_4 + a_5 & 2a_1 - 1/2a_3 + a_4 + a_5 \\ -a_2 + a_4 + a_5 & 2a_1 + a_2 - a_3 - a_4 + a_5 \end{pmatrix} \end{aligned}$$

So

$$\begin{pmatrix} 2a_1 - 2a_2 + 4a_4 + a_5 & 2a_1 - 1/2a_3 + a_4 + a_5 \\ -a_2 + a_4 + a_5 & 2a_1 + a_2 - a_3 - a_4 + a_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is the simplified version of the equation we are trying to solve. Two matrices are equal is they are equal entry-wise, so this gives us the following homogeneous system of linear equations:

$$\begin{array}{rrrrrl} 2a_1 - & 2a_2 & & + 4a_4 & + a_5 & = 0 \\ 2a_1 & & - 1/2a_3 + & a_4 & + a_5 & = 0 \\ & -a_2 & + & a_4 & + a_5 & = 0 \\ 2a_1 + & a_2 - & a_3 - & a_4 & + a_5 & = 0 \end{array}$$

And now we solve:

$$\begin{pmatrix} 2 & -2 & 0 & 4 & 1 \\ 2 & 0 & -1/2 & 1 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ 2 & 1 & -1 & -1 & 1 \end{pmatrix} \underset{\rho_4-\rho 1}{\overset{\rho_2-\rho_1}{\sim}} \begin{pmatrix} 2 & -2 & 0 & 4 & 1 \\ 0 & 2 & -1/2 & -3 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 3 & -1 & -5 & 0 \end{pmatrix}$$

$$\overset{\rho_3 \leftrightarrow -\rho_2}{\sim} \begin{pmatrix} 2 & -2 & 0 & 4 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 2 & -1/2 & -3 & 0 \\ 0 & 3 & -1 & -5 & 0 \end{pmatrix}$$

$$\underset{\rho_4-3\rho_2}{\overset{\rho_3-2\rho_2}{\sim}} \begin{pmatrix} 2 & -2 & 0 & 4 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -1/2 & -1 & 2 \\ 0 & 0 & -1 & -2 & 3 \end{pmatrix}$$

$$\overset{-2\rho_3}{\sim} \begin{pmatrix} 2 & -2 & 0 & 4 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & -1 & -2 & 3 \end{pmatrix}$$

$$\overset{\rho_4+\rho_3}{\sim} \begin{pmatrix} 2 & -2 & 0 & 4 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

We see that a_4 is the only free variable, so the fourth matrix is superfluous. Therefore

$$\text{Span}\left\{ \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1/2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$
$$= \text{Span}\left\{ \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1/2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$