Homework 5 solutions

#1 a) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -1 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

c) $$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{g} \circ \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -1 \end{pmatrix}}_{f} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} = A$$

(order is important here! we must get a 2X2 matrix since $g \circ f$ is $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3 \xrightarrow{g} \mathbb{R}^2$ i.e. $\mathbb{R}^2 \to \mathbb{R}^2$)

d) i. $$\vec{w} = f(\vec{v}) = f\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+(-1) \\ 0 \\ 2\cdot 2-(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

ii. $$g(\vec{w}) = g\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 1+5 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

iii. $$A\vec{v} = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

same! doing f then g really does correspond to matrix multiplication in this case

#2 a) $\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \overset{\rho_2-\rho_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$

$\overset{-\rho_3 \leftrightarrow \rho_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 2 & -2 & -1 & 1 & 0 \end{array}\right) \overset{\rho_3-2\rho_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & 1 & 2 \end{array}\right)$

$\overset{\frac{1}{2}\rho_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1 \end{array}\right) \underset{\rho_2+2\rho_3}{\overset{\rho_1-\rho_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1/2 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1 \end{array}\right)$

$$A^{-1} = \begin{pmatrix} 3/2 & -1/2 & -1 \\ -1 & 1 & 1 \\ -1/2 & 1/2 & 1 \end{pmatrix}$$

check $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 3/2 & -1/2 & -1 \\ -1 & 1 & 1 \\ -1/2 & 1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

b) Let $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$

Then $A\vec{x} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ x+2y-z \\ -y+2z \end{pmatrix}$

So $A\vec{x} = \vec{b}$ is $\begin{pmatrix} x+z \\ x+2y-z \\ -y+2z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}$ OR $\begin{matrix} x+z=2 \\ x+2y-z=0 \\ -y+2z=-4 \end{matrix}$

i.e. $A\vec{x} = \vec{b}$ is what we are trying to solve for $\vec{x}$

We solve by "dividing" by A i.e. multiplying both sides by A^{-1} <u>on the left</u> (this matters

$$\begin{aligned} A\vec{x} &= \vec{b} \\ A^{-1}(A\vec{x}) &= A^{-1}\vec{b} \\ (A^{-1}A)\vec{x} &= A^{-1}\vec{b} \\ I\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

so $\vec{x} = \begin{pmatrix} 3/2 & -1/2 & -1 \\ -1 & 1 & 1 \\ -1/2 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \\ -5 \end{pmatrix}$

check: $x=7$, $y=-6$, $z=-5$ is the solution:

$$7+(-5)=2 \checkmark$$
$$7+2(-6)-(-5)= 7-12+5=0 \checkmark$$
$$-(-6)+2\cdot(-5)= 6-10=-4 \checkmark$$

The solution is unique because the matrix of coefficients A is invertible (all variables are leading $\Rightarrow$ unique solution)

Note: Computing A^{-1} is faster if one must solve many equations $A\vec{x}=\vec{b}_1$, $A\vec{x}=\vec{b}_2$, $A\vec{x}=\vec{b}_3$ etc

otherwise, computing A^{-1} is about the same as solving like we used to; they both involve getting A in reduced echelon form

either

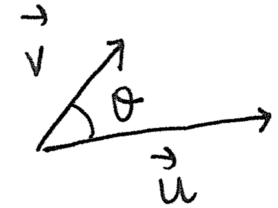
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -1/2 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1 \end{array}\right)$$

OR

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

Actually solving like we used to is a bit faster because $\vec{b}$ is smaller than I_3.

#3 a)



$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}$$

$$= \frac{\begin{pmatrix}3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}}{\sqrt{3^2+1^2}\cdot\sqrt{1^2+2^2}} = \frac{3+2}{\sqrt{10}\cdot\sqrt{5}}$$

$$= \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

b) For any θ, $\cos^2\theta + \sin^2\theta = 1$

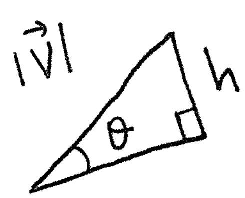
so $\left(\frac{\sqrt{2}}{2}\right)^2 + \sin^2\theta = 1$

$$\frac{2}{4} + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{2}{4}$$

$$\sin\theta = \frac{\sqrt{2}}{2}$$ (not $-\frac{\sqrt{2}}{2}$ since $0 \le \theta \le \frac{\pi}{2}$)

c)



This is a right angle triangle

$$\sin\theta = \frac{\text{opposite}}{\text{hypothenuse}} = \frac{h}{|\vec{v}|}$$

so $h = |\vec{v}|\cdot\sin\theta = \sqrt{1^2+2^2}\cdot\frac{\sqrt{2}}{2} = \sqrt{5}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{10}}{2}$

d) $A = h \cdot b = h \cdot |\vec{u}| = \dfrac{\sqrt{10}}{2} \cdot \sqrt{3^2+1^2} = \dfrac{\sqrt{10} \cdot \sqrt{10}}{2}$

$= \dfrac{10}{2} = 5$

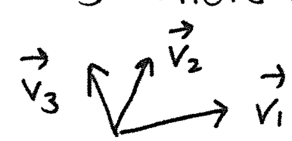
same!

e) $\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 1 \cdot 1 = 6 - 1 = 5$

Remark: This will always be the case.

In $\mathbb{R}^2$, the matrix with columns $\vec{u}$ and $\vec{v}$ $A = (\vec{u}\ \vec{v})$ has determinant the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$ as long as the angle from $\vec{u}$ to $\vec{v}$ is counterclock wise

In $\mathbb{R}^3$, the matrix $A = (\vec{v}_1\ \vec{v}_2\ \vec{v}_3)$ (columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$) has determinant the volume of the parallelopiped with sides $\vec{v}_1, \vec{v}_2, \vec{v}_3$ as long as the 3 vectors respect the right-hand rule (as your fingers on your right hand curl from $\vec{v}_1$ to $\vec{v}_2$, $\vec{v}_3$ sticks out in the direction of your thumb)



Similar things happen in higher dimension.

The determinant is an <u>oriented volume</u>.

#4 a) They are the roots of

$$p(\lambda) = \begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2}-\lambda \end{vmatrix} = \left(\frac{1}{2}-\lambda\right)\left(\frac{5}{2}-\lambda\right) - \left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)$$

$$= \frac{5}{4} - \frac{1}{2}\lambda - \frac{5}{2}\lambda + \lambda^2 + \frac{3}{4}$$

$$= \lambda^2 - 3\lambda + 2$$

$$= (\lambda-1)(\lambda-2)$$

The eigen values are $\lambda_1 = 1$ and $\lambda_2 = 2$.

b) $$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & \frac{5}{2} & 0 & 1 \end{array}\right) \underset{2\rho_2}{\overset{2\rho_1}{\sim}} \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ -3 & 5 & 0 & 2 \end{array}\right)$$

$$\overset{\rho_2+3\rho_1}{\sim} \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 8 & 6 & 2 \end{array}\right) \overset{\frac{1}{8}\rho_2}{\sim} \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} \end{array}\right)$$

$$\overset{\rho_1-\rho_2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & \frac{5}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{3}{4} & \frac{1}{4} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$ check $$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}\begin{pmatrix} \frac{5}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c) They are the roots of

$$p(\lambda) = \begin{vmatrix} \frac{5}{4}-\lambda & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4}-\lambda \end{vmatrix} = \left(\frac{5}{4}-\lambda\right)\left(\frac{1}{4}-\lambda\right) - \left(-\frac{1}{4}\right)\left(\frac{3}{4}\right)$$

$$= \frac{5}{16} - \frac{5}{4}\lambda - \frac{1}{4}\lambda + \lambda^2 + \frac{3}{16}$$

$$= \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$

$$= \frac{1}{2}\left(2\lambda^2 - 3\lambda + 1\right)$$

$$= \frac{1}{2}\left(2\lambda^2 - 2\lambda - \lambda + 1\right)$$

$$= \frac{1}{2}\left(2\lambda(\lambda-1) - 1(\lambda-1)\right) = \frac{1}{2}(\lambda-1)(2\lambda-1)$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$

Note: If A is invertible it will never have eigenvalue $\lambda = 0$ (otherwise there is $\vec{v} \neq \vec{0}$ with $A\vec{v} = 0\vec{v} = \vec{0}$, but if A is invertible the unique solution to $A\vec{v} = \vec{0}$ is $\vec{v} = \vec{0}$)

If A is invertible and its eigenvalues are $\lambda_1, \lambda_2 \ldots \lambda_m$ then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_m}$