Homework 3 Solutions

#1 Since the set of all vectors orthogonal to $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ is contained in $\mathbb{R}^3$, which is known to be a vector space, it suffices to check the subspace property i.e.

if $\vec{u}_1$ and $\vec{u}_2$ are orthogonal to $\vec{v}$ then $r_1\vec{u}_1 + r_2\vec{u}_2$ is also orthogonal to $\vec{v}$, where r_1 and r_2 are real numbers.

Let $\vec{u}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$. If $\vec{u}$ is orthogonal to $\vec{v}$, it means that $\vec{u} \cdot \vec{v} = 0$.

So here
$$\vec{u}_1 \cdot \vec{v} = x_1 + 2y_1 = 0$$
$$\vec{u}_2 \cdot \vec{v} = x_2 + 2y_2 = 0$$

Now compute

$$(r_1\vec{u}_1 + r_2\vec{u}_2) \cdot \vec{v} = \begin{pmatrix} r_1x_1 + r_2x_2 \\ r_1y_1 + r_2y_2 \\ r_1z_1 + r_2z_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$= (r_1x_1 + r_2x_2) + 2(r_1y_1 + r_2y_2)$$
$$= (r_1x_1 + 2r_1y_1) + (r_2x_2 + 2r_2y_2)$$
$$= r_1(x_1 + 2y_1) + r_2(x_2 + 2y_2) = r_1 \cdot 0 + r_2 \cdot 0 = 0.$$

So $r_1\vec{u}_1 + r_2\vec{u}_2$ is orthogonal to $\vec{v}$ and we are done

#2 Let $V = \{ f$ such that $f'' + f = 0 \}$

1. Existence of addition i.e. if $f_1, f_2 \in V$ then $f_1 + f_2 \in V$

f_1 and $f_2 \in V$ means $f_1'' + f_1 = 0$
$f_2'' + f_2 = 0$

Now compute

$$(f_1 + f_2)'' + (f_1 + f_2) = f_1'' + f_2'' + f_1 + f_2$$
$$= (f_1'' + f_1) + (f_2'' + f_2)$$
$$= 0 + 0 = 0$$

So $f_1 + f_2 \in V$

2. Addition is commutative i.e. $f_1 + f_2 = f_2 + f_1$

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \underset{\uparrow}{=} f_2(x) + f_1(x) = (f_2 + f_1)(x)$$

this is true because + in $\mathbb{R}$ is commutative

3. Addition is associative i.e. $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$

$$((f_1 + f_2) + f_3)(x) = (f_1 + f_2)(x) + f_3(x)$$
$$= (f_1(x) + f_2(x)) + f_3(x)$$
$$= f_1(x) + (f_2(x) + f_3(x))$$

← this is true because + in $\mathbb{R}$ is associative

$$= f_1(x) + (f_2 + f_3)(x)$$
$$= (f_1 + (f_2 + f_3))(x)$$

④ Addition has an identity and this identity belongs to V

- addition has an identity:

  let $0 : \mathbb{R} \to \mathbb{R}$, $x \mapsto 0$ i.e. output is always 0

  then $f + 0 = f$:
$$\begin{aligned}(f+0)(x) &= f(x) + 0(x)\\ &= f(x) + 0\\ &= f(x)\end{aligned}$$

  this is true because 0 is the identity for + in $\mathbb{R}$

- this identity belongs to V:

  $0'' + 0 = 0 + 0 = 0$ so $0 \in V$

⑤ Addition has an inverse and this inverse belongs to V

- addition has an inverse:

  for $f : \mathbb{R} \to \mathbb{R}$, let $-f : \mathbb{R} \to \mathbb{R}$, $x \mapsto -f(x)$ i.e. output is $-f(x)$

  then $f + (-f) = 0$:
$$\begin{aligned}(f+(-f))(x) &= f(x) + (-f)(x)\\ &= f(x) - f(x)\\ &= 0\\ &= 0(x)\end{aligned}$$

  this is true because + has an inverse in $\mathbb{R}$

- this inverse belongs to V: Let $f \in V$ i.e. $f'' + f = 0$

  then $(-f)'' + (-f) = -f'' - f = -(f'' + f) = -0 = 0$

  so $-f \in V$

6. Existence of scalar multiplication: i.e. if $f \in V$ then $rf \in V$ for $r \in \mathbb{R}$

$f \in V$ means $f'' + f = 0$

then $(rf)'' + (rf) = rf'' + rf = r(f'' + f) = r \cdot 0 = 0$

so $rf \in V$

7. Addition distributes on scalar multiplication

i.e. $r(f_1 + f_2) = rf_1 + rf_2$

$(r(f_1+f_2))(x) = r(f_1(x) + f_2(x))$ ← this is true because + distributes on multiplication in $\mathbb{R}$

$= rf_1(x) + rf_2(x)$

$= (rf_1 + rf_2)(x)$

8. Multiplication distributes on addition i.e. $(r+s)f = rf + sf$

$((r+s)f)(x) = (r+s)f(x)$ ← this is true because multiplication distributes on + in $\mathbb{R}$

$= rf(x) + sf(x)$

$= (rf + sf)(x)$

9. Scalar multiplication is associative i.e. $r(sf) = (rs)f$

$(r(sf))(x) = r(sf(x))$ ← this is true because multiplication in $\mathbb{R}$ is associative

$= (rs)f(x)$

$= ((rs)f)(x)$

(10) Multiplication by $1 \in \mathbb{R}$ is the identity i.e. $1 \cdot f = f$

$$(1 \cdot f)(x) = 1 \cdot f(x) \quad \leftarrow \text{this is true because}$$
$$= f(x)$$

multiplication by 1 is the identity in $\mathbb{R}$.

Bonus! So $V = \{f \text{ such that } f'' + f = 0\}$ is a vector space!

Facts: its dimension is 2

a basis is $f_1(x) = \sin x$, $f_2(x) = \cos x$

$\Rightarrow$ every $f \in V$ is $f(x) = a_1 \sin x + a_2 \cos x$ for some $a_1, a_2 \in \mathbb{R}$

In general, the solutions of any linear homogeneous differential equation form a vector space.

If the highest derivative that appears is the n^{th} derivative, then the space has dimension n

Example $V = \{f \text{ such that } f''' + f'' + 4f' + 4f = 0\}$

has dimension 3 and basis

$$f_1(x) = \sin(2x) \quad f_2(x) = \cos(2x) \quad f_3(x) = e^{-x}$$