Math 124 - Homework 1 solutions

#1 There are many, many correct solutions to this problem (infinitely many!). Here is one:

a) $$\begin{aligned} x+y&=2 \\ x-y&=0 \end{aligned}$$ has a unique solution

b) $$\begin{aligned} x+y&=2 \\ 2x+2y&=2 \end{aligned}$$ has no solution

c) $$\begin{aligned} x+y&=2 \\ 2x+2y&=4 \end{aligned}$$ has infinitely many solutions

#2 Row reduction is robust enough to solve even the largest, most annoying systems! I'll actually get all the way to reduced echelon form to make my life even easier.

$$\overset{\rho_1 \leftrightarrow \rho_2}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 1 & -1 \\ 3 & -1 & 2 & 4 & 1 & 2 \\ 2 & -3 & 6 & 9 & 4 & -5 \\ 7 & -2 & 4 & 8 & 1 & 6 \end{array}\right] \begin{array}{c} \rho_2 - 3\rho_1 \\ \sim \\ \rho_3 - 2\rho_1 \\ \rho_4 - 7\rho_1 \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 1 & -1 \\ 0 & 2 & -4 & -5 & -2 & 5 \\ 0 & -1 & 2 & 3 & 2 & -3 \\ 0 & 5 & -10 & -13 & -6 & 13 \end{array}\right]$$

$$\overset{-\rho_2 \leftrightarrow \rho_3}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 1 & -1 \\ 0 & 1 & -2 & -3 & -2 & 3 \\ 0 & 2 & -4 & -5 & -2 & 5 \\ 0 & 5 & -10 & -13 & -6 & 13 \end{array}\right] \begin{array}{c} \rho_3 - 2\rho_2 \\ \sim \\ \rho_4 - 5\rho_2 \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 1 & -1 \\ 0 & 1 & -2 & -3 & -2 & 3 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 2 & 4 & -2 \end{array}\right]$$

$$\overset{\rho_4 - 2\rho_3}{\sim} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 1 & -1 \\ 0 & 1 & -2 & -3 & -2 & 3 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This is echelon form. Now I'm going to reduced echelon form it.

$$\begin{array}{c} \rho_1 - 3\rho_3 \\ \sim \\ \rho_2 + 3\rho_3 \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -5 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{\rho_1 + \rho_2}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The leading variables are x_1, x_2 and x_4; x_3 and x_5 are free. I have the equations

$$\begin{aligned} x_1 \quad\quad - x_5 &= 2 \\ x_2 - 2x_3 + 4x_5 &= 0 \\ x_3 &= x_3 \\ x_4 \quad + 2x_5 &= -1 \\ x_5 &= x_5 \end{aligned}$$

In vector form this is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 1 \\ -4 \\ 0 \\ -2 \\ 1 \end{pmatrix} x_5$$

Bonus: check the answer

$\vec{p} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ should be a solution of the system

$$\begin{aligned} 3\cdot 2 - 0 + 0 + 4\cdot(-1) + 0 &= 2 \ \checkmark \\ 2 - 0 + 0 + 3(-1) + 0 &= -1 \ \checkmark \\ 2\cdot 2 - 0 + 0 + 9(-1) + 0 &= -5 \ \checkmark \\ 7\cdot 2 - 0 + 0 + 8(-1) + 0 &= 6 \ \checkmark \end{aligned}$$

$\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_3$ should be a solution of the homogeneous system

$$\begin{aligned} 0 - 2x_3 + 2x_3 + 0 + 0 &= 0 \ \checkmark \\ 0 - 2x_3 + 2x_3 + 0 + 0 &= 0 \ \checkmark \\ 0 - 3(2x_3) + 6x_3 + 0 + 0 &= 0 \ \checkmark \\ 0 - 2(2x_3) + 4x_3 + 0 + 0 &= 0 \ \checkmark \end{aligned}$$

$\begin{pmatrix} 1 \\ -4 \\ 0 \\ -2 \\ 1 \end{pmatrix} x_5$ should be a solution of the homogeneous system

$$\begin{aligned} 3x_5 - (-4x_5) + 0 + 4(-2x_5) + x_5 &= 0 \ \checkmark \\ x_5 - (-4x_5) + 0 + 3(-2x_5) + x_5 &= 0 \ \checkmark \\ 2x_5 - 3(-4x_5) + 0 + 9(-2x_5) + 4x_5 &= 0 \ \checkmark \\ 7x_5 - 2(-4x_5) + 0 + 8(-2x_5) + x_5 &= 0 \ \checkmark \end{aligned}$$

yay!! it's correct!!

#3a) We do this in 2 steps

1. Find which of the vectors are solutions, and therefore can serve as particular solutions
2. Find the solution to the homogeneous system

Then we bring them together as $\vec{p}+H$. There could be multiple $\vec{p}$'s, so we'll get possibly a few different-looking general solutions.

check $\begin{pmatrix}0\\0\\0\\4\end{pmatrix}$:

$$\begin{aligned} 0-0 \quad +4 &= 4 \checkmark \\ 0+0-0 \quad &= 0 \checkmark \\ 0+0+4 &= 4 \checkmark \end{aligned}$$

it's a solution

check $\begin{pmatrix}-5\\1\\-7\\10\end{pmatrix}$:

$$\begin{aligned} -5-1 \quad +10 &= 4 \checkmark \\ 2(-5)+3(1)-(-7) &= 0 \checkmark \\ 1+(-7)+10 &= 4 \checkmark \end{aligned}$$

it's a solution

check $\begin{pmatrix}2\\-1\\1\\1\end{pmatrix}$:

$$\begin{aligned} 2-(-1) \quad +1 &= 4 \checkmark \\ 2\cdot 2+3(-1)-1 &= 0 \checkmark \\ -1+1+1 &= 1 \times \end{aligned}$$

not a solution

We now solve the homogeneous system

$$\begin{aligned} x-y \quad +w &= 0 \\ 2x+3y-z \quad &= 0 \\ y+z+w &= 0 \end{aligned}$$

This is easier than the original system because we won't have to worry about the constants. (Every little bit counts!)

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 3 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \overset{\rho_2 - 2\rho_1}{\sim} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 5 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{bmatrix} \overset{\rho_2 \leftrightarrow \rho_3}{\sim} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & -1 & -2 \end{bmatrix}$$

Row of constants is omitted since all zeroes

Keep going to reduced echelon form

$$\overset{\rho_3 - 5\rho_2}{\sim} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -6 & -7 \end{bmatrix} \overset{-\frac{1}{6}\rho_3}{\sim} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 7/6 \end{bmatrix}$$

$$\overset{\rho_2 - \rho_3}{\sim} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 7/6 \end{bmatrix} \overset{\rho_1 + \rho_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5/6 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 7/6 \end{bmatrix}$$

w is free. The solution is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5/6 \\ 1/6 \\ -7/6 \\ 1 \end{pmatrix} w$$

check:

$$-5/6\,w - 1/6\,w \quad + w = 0 \ \checkmark$$
$$2(-5/6\,w) + 3(1/6\,w) - (-7/6\,w) = 0 \ \checkmark \qquad \text{it's correct!}$$
$$1/6\,w - 7/6\,w + w = 0 \ \checkmark$$

We get two "looks" for the general solution:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -5/6 \\ 1/6 \\ -7/6 \\ 1 \end{pmatrix} w \quad \& \quad \begin{pmatrix} -5 \\ 1 \\ -7 \\ 10 \end{pmatrix} + \begin{pmatrix} -5/6 \\ 1/6 \\ -7/6 \\ 1 \end{pmatrix} w$$

b) There is a value of w such that

$$\begin{pmatrix}0\\0\\0\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\4\end{pmatrix} + \begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix} w = \begin{pmatrix}0-5/6\,w\\0+1/6\,w\\0-7/6\,w\\4+w\end{pmatrix}$$

Another way to put it is:

$$0 = -5/6\,w$$
$$0 = 1/6\,w$$
$$0 = -7/6\,w$$
$$4 = 4 + w$$

For each, the solution is

$$\boxed{w = 0}$$

c) There is a (different!) value of w such that

$$\begin{pmatrix}0\\0\\0\\4\end{pmatrix} = \begin{pmatrix}-5\\1\\-7\\10\end{pmatrix} + \begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix} w$$

Reading each row, this is

$$0 = -5 - 5/6\,w$$
$$0 = 1 + 1/6\,w$$
$$0 = -7 - 7/6\,w$$
$$4 = 10 + w$$

For each, the solution is

$$\boxed{w = -6}$$

d) This time we find w such that

$$\begin{pmatrix}-5\\1\\-7\\10\end{pmatrix} = \begin{pmatrix}0\\0\\0\\4\end{pmatrix} + \begin{pmatrix}-5/6\\1/6\\-7/6\\1\end{pmatrix} w$$

So we solve

$$-5 = -5/6\,w$$
$$1 = 1/6\,w$$
$$-7 = -7/6\,w$$
$$10 = 4 + w$$

$$\boxed{w = 6}$$

e) It's pretty easy to see with all this practice that the solution to

$$\begin{pmatrix} -5 \\ 1 \\ -7 \\ 10 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ -7 \\ 10 \end{pmatrix} + \begin{pmatrix} -5/6 \\ 1/6 \\ -7/6 \\ 1 \end{pmatrix} w \quad \text{is} \quad \boxed{w=0}$$

f) We plug in $w=2$ into each. We'll get 2 different solutions

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -5/6 \\ 1/6 \\ -7/6 \\ 1 \end{pmatrix} 2 = \begin{pmatrix} -5/3 \\ 1/3 \\ -7/3 \\ 6 \end{pmatrix}$$

this really is a sol^{n}!

$-5/3 - 1/3 + 6 = 4$ ✓

$2(-5/3) + 3(1/3) - (-7/3) = 0$ ✓

$1/3 + (-7/3) + 6 = 4$ ✓

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ -7 \\ 10 \end{pmatrix} + \begin{pmatrix} -5/6 \\ 1/6 \\ -7/6 \\ 1 \end{pmatrix} 2 = \begin{pmatrix} -20/3 \\ 4/3 \\ -28/3 \\ 12 \end{pmatrix}$$

this really is a solution!

$-20/3 - (4/3) + 12 = 4$ ✓

$2(-20/3) + 3(4/3) - (-28/3) = 0$ ✓

$4/3 - 28/3 + 12 = 4$ ✓

I hope that you are convinced that even though our 2 general solutions looked different, they are the same because any solution you can get from one you can get from the other (using a different w) and they both give all solutions to the system.