Math 124: Fall 2016
Practice for Final Exam

NAME: SOLUTIONS

Time: **2 hours and 30 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score	Problem	Value	Score
1	5		9	6	
2	6		10	10	
3	4		11	5	
4	4		12	10	
5	4		13	8	
6	4		14	8	
7	6		15	6	
8	8		16	6	
			TOTAL	100	

Problem 1 : (5 points) Let $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a homomorphism.

a) Suppose that f is onto. What is its rank?

"onto" means that the image / range space is all of $\mathbb{R}^4$ i.e. 4-dimensional

the rank is 4

b) Suppose now that f is one-to-one. What is its nullity?

"one-to-one" means in particular that $\vec{0}$ is the only vector such that $f(\vec{0})=\vec{0}$. Since $\{\vec{0}\}$ is 0-dimensional,

the nullity is 0

c) Again, suppose that f is one-to-one. Is it possible to know if f is onto?

we know that dim domain = rank + nullity

If dim domain = 4 & nullity = 0 then rank = 4

so f <u>is</u> onto

d) Suppose now that the rank of f is 3. What is the nullity of f?

dim domain = rank + nullity

$4 = 3 + 1$

so nullity is 1

e) Finally, suppose now that f is an isomorphism. What is the rank of f? What is its nullity?

isomorphism means onto $\rightarrow$ so the rank is 4

and one-to-one $\rightarrow$ so the nullity is 0

Problem 2 : (6 points) For each of the following matrices, say if it is in reduced echelon form, in echelon form only, or neither. For each, say which variables are free and which variables are leading if the first column corresponds to x, the second column to y and the last column to z.

a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ echelon form only
(leading 2 instead of leading 1, but we do have zeroes under each leading position)

x, y, z are all leading, no free variables

b) $\begin{pmatrix} 1 & 9 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ reduced echelon form
(all leading ones and zeroes above and below the leading positions)

x and z are leading
y is free

c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ neither echelon nor reduced echelon
to find leading and free variables we must be at least in echelon form

$$\underset{\rho_3-\rho_1}{\overset{\rho_2-\rho_1}{\sim}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

← this is echelon form (but not reduced!)

x and y are leading
z is free

Problem 3 : (4 points) Solve the following system of linear equations. If you do find solution(s), check your answer.

$$\begin{aligned} x \quad\quad - z &= 1 \\ 2x + y \quad\quad &= 2 \\ 2x + 2y + 2z &= 2 \end{aligned}$$

augmented matrix for the system

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{array}\right) \underset{\rho_3 - 2\rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right)$$

$$\overset{\rho_3 - 2\rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x - z = 1 \rightsquigarrow x = z + 1$

$y + 2z = 0 \rightsquigarrow y = -2z$

z is free

because I'm solving I like this to be reduced echelon form

Solution in vector form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

check particular solution

$x = 1, y = 0, z = 0$:

$1 - 0 = 1$ ✓

$2 + 0 = 2$ ✓

$2 + 0 = 2$ ✓

check homogeneous solution

$x = z, y = -2z, z = z$

$z - z = 0$ ✓

$2z + (-2z) = 0$ ✓

$2z + 2(-2z) + 2z = 0$ ✓

Problem 4 : (4 points) Consider the set

$$\left\{ \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Is this set linearly dependent or linearly independent?

We solve $a_1 \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} + a_2 \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix} + a_3 \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} + a_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

If the only solution is $a_1 = a_2 = a_3 = a_4 = 0$, the set is linearly independent

$$\begin{aligned} 2a_1 - 2a_2 + 4a_3 + a_4 &= 0 \\ 2a_1 \qquad + a_3 + a_4 &= 0 \\ -a_2 + a_3 + a_4 &= 0 \\ 2a_1 + a_2 - a_3 + a_4 &= 0 \end{aligned}$$

matrix of coefficients:

$$\begin{pmatrix} 2 & -2 & 4 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 2 & 1 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow[\rho_4 - \rho_1]{\rho_2 - \rho_1} \begin{pmatrix} 2 & -2 & 4 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -5 & 0 \end{pmatrix} \xrightarrow{-\rho_3 \leftrightarrow \rho_2} \begin{pmatrix} 2 & -2 & 4 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -3 & 0 \\ 0 & 3 & -5 & 0 \end{pmatrix}$$

$$\xrightarrow[\rho_4 - 3\rho_2]{\rho_3 - 2\rho_2} \begin{pmatrix} 2 & -2 & 4 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 3 \end{pmatrix} \xrightarrow{\rho_4 - 2\rho_3} \begin{pmatrix} 2 & -2 & 4 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

just checking for leading variables so echelon only is enough

all variables are leading so solution is unique, the vectors are linearly independent

(or all variables are leading so no vector is superfluous; the vectors are linearly independent)

Problem 5 : (4 points) Consider the homogeneous system of linear equations

$$\begin{aligned} x - \ y + \ z \qquad &= 0 \\ y \qquad + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \qquad - w &= 0 \end{aligned}$$

What is the dimension of its solution set? Support your answer by giving a basis. Be sure to argue that you have found a basis.

We solve, write in vector form, this will give a basis

$$\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 3 & -2 & 3 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \xrightarrow{\rho_3 - 3\rho_1} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \xrightarrow[\rho_4+\rho_2]{\rho_1+\rho_2,\ \rho_3-\rho_2} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

I am solving so I like for this to be reduced echelon form

x, y are leading
z, w are free
} solution set is 2-dimensional (# of free variables)

$x + z + w = 0 \rightsquigarrow x = -z - w$

$y + w = 0 \rightsquigarrow y = -w$

$z = z$

$w = w$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} z + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} w$$

A basis is $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

This set is linearly independent; can be seen by looking at 3rd and 4th coordinates.

Problem 6 : (4 points) What is the dimension of the space

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \quad \text{such that} \quad 2x_3 + x_4 = 0 \right\}?$$

Support your answer by giving a basis for the space. Be sure to argue that you have found a basis.

Same! Solve and write in vector form

$$\begin{pmatrix} 0 & 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & 1/2 \end{pmatrix}$$ this is reduced echelon form!

$\left.\begin{array}{l} x_1, x_2 \text{ and } x_4 \text{ are free} \\ x_3 \text{ is leading} \end{array}\right\}$ The space is 3-dimensional

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_2 \\ x_3 &= -\tfrac{1}{2}x_4 \\ x_4 &= x_4 \end{aligned} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 1 \end{pmatrix} x_4$$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 1 \end{pmatrix} \right\}$ is a basis. for V. These 3 vectors can be seen to be linearly independent by looking at the 1st, 2nd and 4th coordinates.

Problem 7 : (6 points) Perform the following matrix operations if they are defined. If they are not defined, state "not defined."

a) $\begin{pmatrix} 5 & -1 & 2 \\ 6 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

This is not defined because matrix addition is only defined when matrices are the same size.

b) $\begin{pmatrix} 1 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

$2 \times \underbrace{3 \cdot 3}_{\text{same so defined}} \times 3$ answer will be 2×3

$$= \begin{pmatrix} 2+3-3 & -1+1-1 & -1+1-1 \\ 8+9 & -4+3 & -4+3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 17 & -1 & -1 \end{pmatrix}$$

Problem 8 : (8 points) For each of the following matrices, compute the inverse of the matrix, if it exists.

a) $\begin{pmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

Let's try and see what happens!

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \overset{\rho_2-\rho_1}{\underset{\rho_3-\rho_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{\rho_1+\rho_2}{\underset{\rho_3-3\rho_2}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{array}\right) \overset{\rho_1-2\rho_3}{\underset{\rho_2+\rho_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 7 & -2 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{array}\right)$$

inverse exists and is $\begin{pmatrix} -4 & 7 & -2 \\ 1 & -2 & 1 \\ 2 & -3 & 1 \end{pmatrix}$

check your answer by computing

$$\begin{pmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} -4 & 7 & -2 \\ 1 & -2 & 1 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) $\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 2 & 1 & 2 & 4 \\ 1 & 3 & 1 & 2 \end{pmatrix}$

$$\left(\begin{array}{cccc|cccc} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 4 & 0 & 0 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \overset{\rho_3-2\rho_1}{\underset{\rho_4-\rho_1}{\sim}} \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

uh oh...

$$\overset{\rho_4-\frac{4}{3}\rho_3}{\sim} \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5/3 & 0 & -4/3 & 1 \end{array}\right)$$

The last row is all zeroes, so we can never get the identity matrix. The inverse does not exist.

Problem 9 : (6 points) Compute the determinant of each of the following matrices. Decide if the matrix is invertible or not.

a) $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2\cdot 2 - 1\cdot 4 = 4-4=0$$

The matrix is not invertible.

b) $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

expand along this column since it has 3 zeroes

$$\text{determinant} = (-1)^{2+2}\cdot 1 \cdot \begin{vmatrix} 2 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

expand along this row

(3rd row, 1st & 3rd column are also good)

$$= (-1)^{2+1}(-1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + (-1)^{2+2}\cdot 1 \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= +(1\cdot 1 - 1\cdot 1) + (2\cdot 1 - 1\cdot 0) = 0+2=2$$

The matrix is invertible

Problem 10 : (10 points) Consider the matrix

$$\begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix}.$$

a) (4 points) Find all of the eigenvalues of this matrix.

They are the roots of

$$p(\lambda) = \begin{vmatrix} 4-\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = (4-\lambda)(-\lambda) - 4(-1) = -4\lambda + \lambda^2 + 4$$
$$= (\lambda-2)^2$$

The only eigenvalue is $\lambda = 2$

b) (6 points) For each eigenvalue, find a basis for the eigenspace. Is this matrix diagonalizable?

Since $p(\lambda) = (\lambda-2)^2$, the dimension of the eigenspace with eigenvalue $\lambda=2$ is either 1 or 2. Let's see which.

Solve $\begin{pmatrix} 4-2 & 4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ eigenvectors

$$\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix} \overset{-\rho_2 \leftrightarrow \rho_1}{\sim} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$ so $x + 2y = 0$ or $x = -2y$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} y$ A basis for the eigenspace is $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$

(This is linearly independent since a single nonzero vector is linearly independent)

Since the matrix is 2x2 but has a single linearly independent eigenvector in total, it is <u>not</u> diagonalizable

Problem 11 : (5 points) Give a triple of numbers a, b and c such that the system

$$\begin{aligned} x \quad\quad - z &= a \\ 2x + \ y \quad\quad &= b \\ 2x + 2y + 2z &= c \end{aligned}$$

does not have a solution.

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 2 & 1 & 0 & b \\ 2 & 2 & 2 & c \end{array}\right) \underset{\rho_3 - 2\rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 1 & 2 & b-2a \\ 0 & 2 & 4 & c-2a \end{array}\right)$$

$$\overset{\rho_3 - 2\rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 1 & 2 & b-2a \\ 0 & 0 & 0 & c+2a-2b \end{array}\right)$$

Side work

$$\begin{aligned} &(c-2a)-2(b-2a) \\ &= c-2a-2b+4a \\ &= c+2a-2b \end{aligned}$$

The last row is a contradiction if $c+2a-2b \neq 0$

Any such triple is a correct answer to this question.

Some examples:

$a=1, b=1, c=1$ (or any $c \neq 0$ here)

$a=-1, c=2, b=5$ (or any $b \neq 0$)

etc etc

Problem 12 : (10 points) Consider the following set of vectors:

$$S = \left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ -9 \end{pmatrix} \right\}$$

a) (2 points) Without doing any complicated computations, you should be able to argue that these vectors are not linearly independent. Give the argument.

These are 5 vectors inside $\mathbb{R}^3$, which is 3-dimensional. This means that in $\mathbb{R}^3$, the largest linearly independent set has 3 elements. This is too many so they must be dependent

b) (4 points) Shrink this set to a linearly independent subset T that spans the same space.

$$\begin{pmatrix} -1 & -2 & 1 & -2 & 7 \\ 0 & 0 & 1 & 1 & -3 \\ 3 & 6 & -2 & 2 & -9 \end{pmatrix} \underset{\rho_3+3\rho_1}{\overset{-\rho_1}{\sim}} \begin{pmatrix} 1 & 2 & -1 & 2 & 7 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -4 & 12 \end{pmatrix}$$

$$\overset{\rho_3-\rho_2}{\sim} \begin{pmatrix} 1 & 2 & -1 & 2 & 7 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & -5 & 15 \end{pmatrix}$$

leading

All we need are leading variables so echelon form is enough

$$T = \left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

This is guaranteed to be linearly independent.

c) (4 points) For $\vec{v} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, give $\mathrm{Rep}_T(\vec{v})$.

Note that T is a basis for $\mathbb{R}^3$ since it is 3 linearly independent vectors

Solve $a_1 \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + a_3 \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

here I go to Reduced echelon because I am solving

$$\left(\begin{array}{ccc|c} -1 & 1 & -2 & 3 \\ 0 & 1 & 1 & 1 \\ 3 & -2 & 2 & 2 \end{array}\right) \underset{\rho_3+3\rho_1}{\overset{-\rho_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -4 & 11 \end{array}\right) \underset{\rho_3-\rho_2}{\overset{\rho_1+\rho_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 10 \end{array}\right) \overset{-\frac{1}{5}\rho_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\mathrm{Rep}_T(\vec{v}) = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \begin{array}{l} \leftarrow a_1 \\ \leftarrow a_2 \\ \leftarrow a_3 \end{array}$$

plug in & check in original equation!

Problem 13 : (8 points)

a) Is the vector $\begin{pmatrix}1\\3\end{pmatrix}$ in the column space of the matrix $\begin{pmatrix}1&1\\1&1\end{pmatrix}$?

Is there a solution for $a_1\begin{pmatrix}1\\1\end{pmatrix}+a_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$?

Augmented matrix: $\left(\begin{array}{cc|c}1&1&1\\1&1&3\end{array}\right)\sim\left(\begin{array}{cc|c}1&1&1\\ \underbrace{0&0&2}\end{array}\right)$

$0=2$ contradiction
no solution

No it is not in the column space.

b) Is the vector $\begin{pmatrix}1\\0\end{pmatrix}$ in the row space of the matrix $\begin{pmatrix}2&1\\3&1\end{pmatrix}$?

Is there a solution for $a_1\begin{pmatrix}2\\1\end{pmatrix}+a_2\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$?

Augmented matrix

$$\left(\begin{array}{cc|c}2&3&1\\1&1&0\end{array}\right)\sim\left(\begin{array}{cc|c}1&1&0\\2&3&1\end{array}\right)\sim\left(\begin{array}{cc|c}1&1&0\\0&1&1\end{array}\right)\sim\left(\begin{array}{cc|c}1&0&-1\\0&1&1\end{array}\right)$$

yes : $a_1=-1$, $a_2=1$

Yes it is in the row space

(check that

$-\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ ✓)

Problem 14 : (8 points) Consider the homomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ whose matrix representation is

$$\begin{pmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \end{pmatrix}.$$

a) Give a basis for the null space of this homomorphism. Be sure to argue that you have found a basis.

Null space is $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $\begin{pmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

matrix of coefficients:

$$\begin{pmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\left.\begin{array}{l} x \text{ is leading} \\ y, z \text{ are free} \end{array}\right\}$ nullity is 2

$2x + 3z = 0 \rightsquigarrow x = -\frac{3}{2}z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix} z$$

Basis: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 0 \\ 1 \end{pmatrix} \right\}$

This is seen to be linearly independent by looking at 2^{nd} and 3^{rd} coord.

b) Give a basis for the range space of this homomorphism. Be sure to argue that you have found a basis.

Range space = column space of matrix

shrink $\left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix} \right\}$ to a basis

Note: since

dim domain = rank + null

$3 = \text{rank} + 2$

we should get 1 basis vector.

$$\begin{pmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

↑ leading

Basis: $\left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$

Our process guarantees this is a basis.

Problem 15 : (6 points) Consider the map

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} y+3z \\ 2x+3y+4z \\ -2x-y+z \end{pmatrix}.$$

Prove that f is an isomorphism.

f is represented by the matrix $\begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 4 \\ -2 & -1 & 1 \end{pmatrix}$

If this matrix is invertible, f is an isomorphism.

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 2 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{\rho_3+\rho_1} \left(\begin{array}{ccc|ccc} 2 & 3 & 4 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 0 & 1 & 1 \end{array}\right) \underset{\rho_3-2\rho_2}{\overset{\rho_1-3\rho_2}{\sim}} \left(\begin{array}{ccc|ccc} 2 & 0 & -5 & -3 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right)$$

$$\begin{array}{c} -\rho_3 \\ \sim \\ \rho_1-5\rho_3 \\ \rho_2+3\rho_3 \end{array} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & -4 & -5 \\ 0 & 1 & 0 & -5 & 3 & 3 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right) \overset{\frac{1}{2}\rho_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & -2 & -5/2 \\ 0 & 1 & 0 & -5 & 3 & 3 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right)$$

Matrix is invertible so f is an isomorphism.

Problem 16 : (6 points) In this problem, consider the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$, with the usual function addition and scalar multiplication.

a) Prove that function addition is associative.

We show that $(f_1+f_2)+f_3 = f_1+(f_2+f_3)$ by showing that both functions take the same value for all x

$$((f_1+f_2)+f_3)(x) = (f_1+f_2)(x)+f_3(x) = (f_1(x)+f_2(x))+f_3(x)$$
$$= f_1(x)+(f_2(x)+f_3(x)) = f_1(x)+(f_2+f_3)(x)$$
$$= (f_1+(f_2+f_3))(x).$$

b) Assuming that the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ with the usual function addition and scalar multiplication is a vector space, prove that the subset

$$S = \{f\colon \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad f'' + f = 0\}$$

is a subspace of the vector space of all functions.

Since S is inside a known vector space, it is enough to show the subspace property:

Let f_1 and $f_2 \in S$, i.e. $f_1''+f_1=0$
$f_2''+f_2=0$

Then does $r_1f_1+r_2f_2$, for $r_1,r_2 \in \mathbb{R}$, also belong to S?

$$(r_1f_1+r_2f_2)''+(r_1f_1+r_2f_2)$$
$$= r_1f_1''+r_2f_2''+r_1f_1+r_2f_2$$
$$= r_1(f_1''+f_1)+r_2(f_2''+f_2)$$
$$= r_1\cdot 0+r_2\cdot 0 = 0+0=0$$

yes, $r_1f_1+r_2f_2 \in S$, so
S is a subspace