Math 124: Fall 2016
Practice for Final Exam

NAME:

Time: **2 hours and 30 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ____

Problem	Value	Score	Problem	Value	Score
1	5		9	6	
2	6		10	10	
3	4		11	5	
4	4		12	10	
5	4		13	8	
6	4		14	8	
7	6		15	6	
8	8		16	6	
			TOTAL	100	

Problem 1 : (5 points) Let $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a homomorphism.

a) Suppose that f is onto. What is its rank?

b) Suppose now that f is one-to-one. What is its nullity?

c) Again, suppose that f is one-to-one. Is it possible to know if f is onto?

d) Suppose now that the rank of f is 3. What is the nullity of f?

e) Finally, suppose now that f is an isomorphism. What is the rank of f? What is its nullity?

Problem 2 : (6 points) For each of the following matrices, say if it is in reduced echelon form, in echelon form only, or neither. For each, say which variables are free and which variables are leading if the first column corresponds to x, the second column to y and the last column to z.

a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 9 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Problem 3 : (4 points) Solve the following system of linear equations. If you do find solution(s), check your answer.

$$\begin{aligned} x \qquad - \ z &= 1 \\ 2x + \ y \qquad &= 2 \\ 2x + 2y + 2z &= 2 \end{aligned}$$

Problem 4 : (4 points) Consider the set

$$\left\{ \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Is this set linearly dependent or linearly independent?

Problem 5 : (4 points) Consider the homogeneous system of linear equations

$$\begin{aligned} x - y + z \phantom{{}+w} &= 0 \\ y \phantom{{}+3z} + w &= 0 \\ 3x - 2y + 3z + w &= 0 \\ -y \phantom{{}+3z} - w &= 0 \end{aligned}$$

What is the dimension of its solution set? Support your answer by giving a basis. Be sure to argue that you have found a basis.

Problem 6 : (4 points) What is the dimension of the space

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \quad \text{such that} \quad 2x_3 + x_4 = 0 \right\}?$$

Support your answer by giving a basis for the space. Be sure to argue that you have found a basis.

Problem 7 : (6 points) Perform the following matrix operations if they are defined. If they are not defined, state "not defined."

a) $\begin{pmatrix} 5 & -1 & 2 \\ 6 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Problem 8 : (8 points) For each of the following matrices, compute the inverse of the matrix, if it exists.

a) $\begin{pmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 2 & 1 & 2 & 4 \\ 1 & 3 & 1 & 2 \end{pmatrix}$

Problem 9 : (6 points) Compute the determinant of each of the following matrices. Decide if the matrix is invertible or not.

a) $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

Problem 10 : (10 points) Consider the matrix

$$\begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix}.$$

a) (4 points) Find all of the eigenvalues of this matrix.

b) (6 points) For each eigenvalue, find a basis for the eigenspace. Is this matrix diagonalizable?

Problem 11 : (5 points) Give a triple of numbers a, b and c such that the system

$$\begin{aligned} x \qquad - \ \ z &= a \\ 2x + \ \ y \qquad &= b \\ 2x + 2y + 2z &= c \end{aligned}$$

does not have a solution.

Problem 12 : (10 points) Consider the following set of vectors:

$$S = \left\{ \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ -9 \end{pmatrix} \right\}$$

a) (2 points) Without doing any complicated computations, you should be able to argue that these vectors are not linearly independent. Give the argument.

b) (4 points) Shrink this set to a linearly independent subset T that spans the same space.

c) (4 points) For $\vec{v} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, give $\mathrm{Rep}_T(\vec{v})$.

Problem 13 : (8 points)

a) Is the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ in the column space of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

b) Is the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the row space of the matrix $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$?

Problem 14 : (8 points) Consider the homomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ whose matrix representation is

$$\begin{pmatrix} 2 & 0 & 3 \\ 4 & 0 & 6 \end{pmatrix}.$$

a) Give a basis for the null space of this homomorphism. Be sure to argue that you have found a basis.

b) Give a basis for the range space of this homomorphism. Be sure to argue that you have found a basis.

Problem 15 : (6 points) Consider the map

$$\begin{aligned} f\colon \mathbb{R}^3 &\to \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} &\mapsto \begin{pmatrix} y+3z \\ 2x+3y+4z \\ -2x-y+z \end{pmatrix}. \end{aligned}$$

Prove that f is an isomorphism.

Problem 16 : (6 points) In this problem, consider the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$, with the usual function addition and scalar multiplication.

a) Prove that function addition is associative.

b) Assuming that the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ with the usual function addition and scalar multiplication is a vector space, prove that the subset

$$S = \{f\colon \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad f'' + f = 0\}$$

is a subspace of the vector space of all functions.