Math 124: Fall 2016
Practice for Exam 2

NAME: SOLUTIONS

Time: **1 hour and 15 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	3	
2	4	
3	4	
4	6	
5	6	
6	5	
7	5	
8	5	
9	12	
TOTAL	50	

Problem 1 : (3 points)

a) What is the dimension of $\mathcal{M}_{4\times4}$? You do not need to justify your answer.

$$4 \cdot 4 = 16$$

b) There is a unique value of k such that $\mathcal{M}_{4\times4}$ is isomorphic to $\mathbb{R}^k$. What is this value of k? You do not need to justify your answer.

$$k=16$$

c) Now let m and n be two positive whole numbers. For what k is $\mathbb{R}^k$ isomorphic to $\mathcal{M}_{m\times n}$?

$$k=mn$$

d) There is a unique value of k such that $\mathcal{P}_5$ is isomorphic to $\mathbb{R}^k$. What is this value of k? You do not need to justify your answer.

$$k=6$$

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5$$

is $\underline{\underline{6}}$ choices to make.

Problem 2 : (4 points) Consider the vector space $\mathcal{P}_2$. For each set S, determine if S is a basis for $\mathcal{P}_2$.

a) $S = \{x^2 - x + 1, 2x + 1, 2x - 1\}$

We use the following fact: If S is a basis for $\mathcal{P}_2$, then <u>every</u> Ax^2+Bx+C in $\mathcal{P}_2$ can be written <u>uniquely</u> as a linear combination of elements of S. (Theorem Two.III.1.12)

Fix A, B, C and solve

$$a_1(x^2-x+1)+a_2(2x+1)+a_3(2x-1) = Ax^2+Bx+C$$

$$a_1x^2+(-a_1+2a_2+2a_3)x+(a_1+a_2-a_3) = Ax^2+Bx+C$$

So
$$a_1 = A$$
$$-a_1+2a_2+2a_3 = B$$
$$a_1+a_2-a_3 = C$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ -1 & 2 & 2 & B \\ 1 & 1 & -1 & C \end{array}\right) \overset{\rho_2+\rho_1,\ \rho_3-\rho_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 2 & 2 & A+B \\ 0 & 1 & -1 & C-A \end{array}\right)$$

$$\overset{\rho_2\leftrightarrow\rho_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & -1 & C-A \\ 0 & 2 & 2 & A+B \end{array}\right) \overset{\rho_3-2\rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & -1 & C-A \\ 0 & 0 & 4 & 3A+B-2C \end{array}\right)$$

see last page for end of problem

b) $S = \{x + x^2, x - x^2, x + 2x^2\}$

Same deal:

$$a_1(x+x^2)+a_2(x-x^2)+a_3(x+2x^2) = Ax^2+Bx+C$$

$$(a_1-a_2+2a_3)x^2+(a_1+a_2+a_3)x = Ax^2+Bx+C$$

So
$$a_1-a_2+2a_3 = A$$
$$a_1+a_2+a_3 = B$$
$$0 = C$$

This is a contradiction if $C \neq 0$. S is not a spanning set so it is not a basis.

Problem 3 : (4 points) Let

$$\vec{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Represent $\vec{v}$ with respect to the following two bases:

a) $B_1 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ We want a_1 & a_2 such that

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solve for a_1 & a_2: $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ -1 & 1 & -1 \end{array}\right) \overset{\rho_2+\rho_1}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & 2 \end{array}\right) \overset{\frac{1}{2}\rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right)$

$\overset{\rho_1-\rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$ $\quad a_1 = 2$, $a_2 = 1$

So $\mathrm{Rep}_{\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

b) $B_2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ We want a_1 & a_2 such that

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Solve: $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 3 & -1 \end{array}\right) \overset{\rho_2-2\rho_1}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -7 \end{array}\right) \overset{\rho_1-\rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & -7 \end{array}\right)$ $\quad a_1 = 10$, $a_2 = -7$

So $\mathrm{Rep}_{\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -7 \end{pmatrix}$

Problem 4 : (6 points) Consider the isomorphism $f\colon \mathcal{P}_1 \to \mathbb{R}^2$ with

$$a + bx \mapsto \mathrm{Rep}_{\{1,1+x\}}(a + bx).$$

Find the image of each of the following elements:

a) $3 - 2x$ We need a_1 & a_2 such that

$$3-2x = a_1 \cdot 1 + a_2(1+x)$$
$$= (a_1+a_2) + a_2 x$$

$$\begin{cases} a_1+a_2=3 \\ a_2=-2 \end{cases}$$

$a_1+(-2)=3$

so $a_1=5$

$$\mathrm{Rep}_{\{1,1+x\}}(3-2x) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

b) $2 + 2x$ We need a_1 & a_2 such that

$$2+2x = a_1 \cdot 1 + a_2(1+x)$$
$$= (a_1+a_2) + a_2 x$$

$$\begin{cases} a_1+a_2=2 \\ a_2=2 \end{cases}$$

$a_1+2=2$

so $a_1=0$

$$\mathrm{Rep}_{\{1,1+x\}}(2+2x) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

c) x We need a_1 & a_2 such that

$$x = a_1 \cdot 1 + a_2(1+x)$$
$$= (a_1+a_2) + a_2 x$$

$$\begin{cases} a_1+a_2=0 \\ a_2=1 \end{cases}$$

$a_1+1=0$

So $a_1=-1$

$$\mathrm{Rep}_{\{1,1+x\}}(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Problem 5 : (6 points) For each pair of a vector $\vec{v}$ and a matrix A, decide if $\vec{v}$ is in the row space of A.

a) $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ -1 & 2 & 7 \end{pmatrix}$

ROWS: $\vec{r}_1 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$ $\vec{r}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\vec{r}_3 = \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix}$

Is there a_1, a_2 & a_3 with $a_1 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

Solve: $\left(\begin{array}{ccc|c} 0 & -1 & -1 & 1 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 7 & 1 \end{array}\right) \overset{\rho_1 \leftrightarrow \rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 3 & 1 & 7 & 1 \end{array}\right) \overset{\rho_3 - 3\rho_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right)$

$\overset{\rho_3 + \rho_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$ this is $0 = -1$, a contradiction

No, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is not in the Row space

b) $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$.

ROWS: $\vec{r}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\vec{r}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Is there a_1, a_2 with $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$?

Solve $\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & 1 & 0 \end{array}\right) \overset{\rho_1 \leftrightarrow \rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 1 \end{array}\right) \overset{\rho_2 - 2\rho_1}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right)$

$\overset{\rho_1 - \rho_2}{\sim} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$ $a_1 = -1$
$a_2 = 1$

Yes, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in the Row space

Problem 6 : (5 points) Give the dimension of the solution set of the homogeneous system of linear equations

$$\begin{aligned} x + y + 2z &= 0 \\ 2x - y + z &= 0 \\ 4x + y + 5z &= 0 \end{aligned}$$

For credit you must give some mathematical justification for your answer.

Solve the system:

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 1 & 5 \end{pmatrix} \underset{\rho_3 - 4\rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{pmatrix} \overset{\rho_3 - \rho_2}{\sim} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\overset{-\frac{1}{3}\rho_2}{\sim} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \overset{\rho_1 - \rho_2}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} x &= -z \\ y &= -z \end{aligned}$$

Solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} z$

The dimension of the solution space is 1.

Justification: The solution set is <u>spanned</u> by the vector $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Since this is a single non zero vector, it is <u>linearly independent</u>. Therefore it is a basis.

Problem 7 : (5 points) Find a basis for the row space of the matrix

$$A = \begin{pmatrix} 2 & 0 & 3 & 4 \\ 0 & 1 & 1 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 0 & -4 & -1 \end{pmatrix}.$$

Make sure to argue that you have found a basis.

Solution like in class: Look at column space of A^T

$$A^T = \begin{pmatrix} 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 3 & 1 & 0 & -4 \\ 4 & -1 & 2 & -1 \end{pmatrix} \underset{\rho_4 - 2\rho_2}{\overset{\rho_3 - \frac{3}{2}\rho_1}{\sim}} \begin{pmatrix} 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -9/2 & -11/2 \\ 0 & -1 & -4 & -3 \end{pmatrix} \underset{\rho_4 + \rho_2}{\overset{\rho_3 - \rho_2}{\sim}} \begin{pmatrix} 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -11/2 & -11/2 \\ 0 & 0 & -3 & -3 \end{pmatrix}$$

$$\underset{-\frac{1}{3}\rho_4}{\overset{-\frac{2}{11}\rho_3}{\sim}} \begin{pmatrix} 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \overset{\rho_4 - \rho_3}{\sim} \begin{pmatrix} 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

echelon form
leading variables in
columns 1,2,3
variable is free in
column 4

The fourth column of A^T / row of A is superfluous; a basis for the row space of A is

$$\{(2034)^T, (011{-1})^T, (3102)^T\}$$

we know this is a basis because this is guaranteed by our process

#7 Alternate solution: Book method
we find a basis for the row space of A directly

$$A=\begin{pmatrix}2&0&3&4\\0&1&1&-1\\3&1&0&2\\1&0&-4&-1\end{pmatrix}\overset{\rho_1\leftrightarrow\rho_4}{\sim}\begin{pmatrix}1&0&-4&-1\\0&1&1&-1\\3&1&0&2\\2&0&3&4\end{pmatrix}\underset{\rho_4-2\rho_1}{\overset{\rho_3-3\rho_1}{\sim}}\begin{pmatrix}1&0&-4&-1\\0&1&1&-1\\0&1&12&5\\0&0&11&6\end{pmatrix}$$

$$\overset{\rho_3-\rho_2}{\sim}\begin{pmatrix}1&0&-4&-1\\0&1&1&-1\\0&0&11&6\\0&0&11&6\end{pmatrix}\overset{\rho_4-\rho_3}{\sim}\begin{pmatrix}1&0&-4&-1\\0&1&1&-1\\0&0&11&6\\0&0&0&0\end{pmatrix}$$ echelon form

A basis for the row space of A is

$$\{(1\ 0\ -4\ -1)^T, (0\ 1\ 1\ -1)^T, (0\ 0\ 11\ 6)^T\}$$

This is a basis because 2 matrices that are row equivalent have the same row space and the rows of a matrix in echelon form are linearly independent.

Note: This is not the same answer as before but that's ok. A space has infinitely many different bases.

Problem 8 : (5 points) Give a basis for the subspace of $\mathcal{P}_2$ given by

$$V = \{a_2x^2 + a_1x + a_0 : a_2 - 2a_1 = a_0\}.$$

Make sure to argue that you have found a basis.

Substitute a_0 into the polynomial

$$= a_2x^2 + a_1x + a_2 - 2a_1$$

Separate the letters

$$= (a_2x^2 + a_2) + (a_1x - 2a_1)$$

Factor out the letters

$$= a_2(x^2+1) + a_1(x-2)$$

$\{x^2+1, x-2\}$ is a spanning set for V by our work above

Now we check for linear independence / shrink if necessary: Suppose that

$$a_1(x^2+1) + a_2(x-2) = 0$$

$$a_1x^2 + a_2x + (a_1 - 2a_2) = 0$$

$$\Rightarrow \left.\begin{array}{l} a_1 = 0 \\ a_2 = 0 \end{array}\right\} \text{only solution}$$

$$a_1 - 2a_2 = 0$$

$\{x^2+1, x-2\}$ is a basis for V

Problem 9 : (12 points) Consider the subspace

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = 3z, y = -z, z \in \mathbb{R} \right\}.$$

a) (3 points) Give a spanning set for V.

Substitute x & y into the vector: $\begin{pmatrix} 3z \\ -z \\ z \end{pmatrix}$

Factor out the letter: $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} z$

$\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a spanning set

b) (3 points) Give a basis for V. Prove that you found a basis.

A single non zero vector is linearly independent.

$\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis

c) (6 points) Show that V is isomorphic to $\mathbb{R}$.
You can use the next page for your work if you run out of space here.

We need a map $f: V \to \mathbb{R}$ which we can show to be an isomorphism. This is provided by $\text{Rep}_{\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

To be precise: $f: V \to \mathbb{R}$

$$\begin{pmatrix} 3z \\ -z \\ z \end{pmatrix} \mapsto z$$

work continued on following page

Please use this page if you need extra space for any problem. (On the problem page, be sure to let me know to look here, and label each problem clearly if you work on multiple problems here.)

#2a) $\overset{\frac{1}{4}\rho_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & -1 & C-A \\ 0 & 0 & 1 & \frac{3}{4}A+\frac{1}{4}B-\frac{1}{2}C \end{array}\right)$

$\overset{\rho_2+\rho_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & \frac{1}{2}C-\frac{1}{4}A+\frac{1}{4}B \\ 0 & 0 & 1 & \frac{3}{4}A+\frac{1}{4}B-\frac{1}{2}C \end{array}\right)$

unique solution

$a_1 = A$

$a_2 = \frac{1}{2}C - \frac{1}{4}A + \frac{1}{4}B$

$a_3 = \frac{3}{4}A + \frac{1}{4}B - \frac{1}{2}C$

Since any Ax^2+Bx+C can be written as a linear combination of elements of S in a unique way, S is a basis.

#9 • f is a homomorphism:

$$f\left(r_1\begin{pmatrix}3z_1\\-z_1\\z_1\end{pmatrix}+r_2\begin{pmatrix}3z_2\\-z_2\\z_2\end{pmatrix}\right) = f\begin{pmatrix}3r_1z_1+3r_2z_2\\-r_1z_1-r_2z_2\\r_1z_1+r_2z_2\end{pmatrix} = r_1z_1+r_2z_2$$

$$r_1 f\begin{pmatrix}3z_1\\-z_1\\z_1\end{pmatrix} + r_2 f\begin{pmatrix}3z_2\\-z_2\\z_2\end{pmatrix} = r_1z_1+r_2z_2 \quad \checkmark$$

• f is one-to-one:

Suppose $f\begin{pmatrix}3z_1\\-z_1\\z_1\end{pmatrix} = f\begin{pmatrix}3z_2\\-z_2\\z_2\end{pmatrix}$ i.e. $z_1 = z_2$

work continued on following page

If $z_1 = z_2$, $\begin{pmatrix} 3z_1 \\ -z_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 3z_2 \\ -z_2 \\ z_2 \end{pmatrix}$ ✓

- f is onto

For any $z \in \mathbb{R}$, the vector $\begin{pmatrix} 3z \\ -z \\ z \end{pmatrix} \in V$ is

such that $f\begin{pmatrix} 3z \\ -z \\ z \end{pmatrix} = z$

we know this because V has all vectors of this form inside of it. So any choice of last entry is in there.