Math 124: Fall 2016
Exam 1

NAME: SOLUTIONS

Time: **50 minutes**

For each problem, you **must** write down all of your work carefully and legibly to receive full credit. For each question, you **must** use theorems and/or mathematical reasoning to support your answer, as appropriate.

Failure to follow these instructions will constitute a breach of the UVM Code of Academic Integrity:

- You may not use a calculator or any notes or book during the exam.
- You may not access your cell phone during the exam for any reason; if you think that you will want to check the time please wear a watch.
- The work you present must be your own.
- Finally, you will more generally be bound by the UVM Code of Academic Integrity, which stipulates among other things that you may not communicate with anyone other than the instructor during the exam, or look at anyone else's solutions.

I understand and accept these instructions.

Signature: ______________________________

Problem	Value	Score
1	6	
2	6	
3	13	
4	6	
5	6	
6	6	
7	7	
TOTAL	50	

Problem 1 : (6 points) Give the reduced echelon form of the following matrix:

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ -1 & -3 & -3 \end{pmatrix}$$

$$\underset{\rho_3+\rho_1}{\overset{\rho_2-2\rho_1}{\sim}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & -6 & 2 \\ 0 & 0 & -2 \end{pmatrix} \overset{\rho_2+\rho_3}{\sim} \begin{pmatrix} 1 & 3 & 1 \\ 0 & -6 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\underset{-\frac{1}{2}\rho_3}{\overset{-\frac{1}{6}\rho_2}{\sim}} \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\rho_1-\rho_3}{\sim} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\rho_1-3\rho_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2 : (6 points)
Find all value(s) k such that the two vectors

$$\begin{pmatrix} k \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

are perpendicular.

These two vectors are perpendicular if their dot product is 0.

$$\begin{pmatrix} k \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4k+3$$

$4k+3=0$ when $4k=-3$

$k=-3/4$

This is the only value of k making these perpendicular.

Problem 3 : (13 points)

a) (5 points) Solve the following system of linear equations:

$$\begin{aligned} x \quad\quad - z \quad\quad &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 1 & 2 & 3 & -1 & 7 \end{array}\right) \overset{\rho_3-\rho_1}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 2 & 4 & -1 & 6 \end{array}\right)$$

$$\overset{\rho_3-2\rho_2}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{\rho_2+\rho_3}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

z is free, others are leading

$x = z + 1$

$y = -2z + 3$

$z = z$

$w = 0$

b) (4 points) Now solve this system of linear equations.
Hint: This is the homogeneous system associated to the system in part a).

$$\begin{aligned} x \quad\; - z \quad\;\; &= 0 \\ y + 2z - w &= 0 \\ x + 2y + 3z - w &= 0 \end{aligned}$$

Writing our previous solution in vector form we get

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z}_{\text{homogeneous part}} + \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

So the solution to this system is

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} z \qquad \text{OR} \qquad \begin{aligned} x &= z \\ y &= -2z \\ z &= z \\ w &= 0 \end{aligned}$$

c) (4 points) Check your answer to part a). (This should look like the solutions to Homework 1 or the solutions to the Practice Exam 1.)
For your convenience, the system you had to solve was:

$$\begin{aligned} x \qquad - z \qquad &= 1 \\ y + 2z - w &= 3 \\ x + 2y + 3z - w &= 7 \end{aligned}$$

Check the particular solution:

$1 - 0 = 1$ ✓
$3 + 0 = 3$ ✓
$1 + 2 \cdot 3 + 0 = 7$ ✓

$\vec{p}$ is correct.

to check the homogeneous solution, it suffices to check the vector that spans the solution set:

$1 - 1 = 0$ ✓
$-2 + 2 \cdot 1 = 0$ ✓
$1 + 2(-2) + 3 \cdot 1 = 0$ ✓

H is correct.

Problem 4 : (6 points) Consider the set

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

Is this set linearly dependent or linearly independent?

What are the solutions to

$$a_1 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ?$$

We solve the homogeneous system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 4 \end{pmatrix} \overset{\rho_3 \to \rho_1 \to \rho_2}{\sim} \begin{pmatrix} -1 & 4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \overset{-\rho_1}{\sim} \begin{pmatrix} 1 & -4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

both a_1 and a_2 are leading variables, therefore we have the unique solution $a_1 = a_2 = 0$ and the set is

linearly independent

Problem 5 : (6 points) Is the vector

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

in the set

$$\text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}?$$

Is there a_1 and a_2 with

$$a_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad ?$$

we solve the system:

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 3 \end{array}\right) \overset{\rho_1 \leftrightarrow \rho_2}{\sim} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 1 \\ -1 & 1 & 3 \end{array}\right) \underset{\rho_3 + \rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{array}\right)$$

This last row is $0 = 3$, a contradiction

Therefore there are no such a_1 and a_2

and $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ is <u>not</u> in $\text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

Problem 6 : (6 points) Give a set that spans the subspace

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 3x + 2y + z = 0 \right\}$$

of the vector space $\mathbb{R}^3$.
Hint: Pretend you are solving the linear equation $3x + 2y + z = 0$.

In the single equation $3x+2y+z=0$, x is leading and y and z are free.

Writing x in terms of y and z we get

$$x = \frac{1}{3}(-2y-z) = -\frac{2}{3}y - \frac{1}{3}z$$

In vector form this is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} z$$

Therefore this subspace is spanned by the set

$$\left\{ \begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Problem 7 : (7 points) You have shown in suggested problems that the set of positive real numbers $\mathbb{R}^+$ is a vector space when we interpret $x \oplus y$ to mean the product of x and y (so that $2 \oplus 3 = 2 \cdot 3 = 6$), and we interpret $r \otimes x$ as the r-th power of x (in other words, $r \otimes x = x^r$).

a) (3 points) Perform the following operations, **with their meaning as above.**

i. $3 \oplus 4$ $= 3 \cdot 4 = 12$

ii. $3 \otimes 2$ $= 2^3 = 8$

iii. $-\frac{1}{2} \otimes 4$ $= 4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{2}$

b) (2 points) In this vector space, what is the zero vector? (For partial credit you may write down the axiom giving the property of the zero vector.)

Axiom: There exists $\vec{0} \in V$ with $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v} \in V$.

Let z be the "zero" value, $x \in \mathbb{R}^+$, and solve $x \oplus z = x$ for z;

$x \oplus z = xz = x$ so $\boxed{z = 1}$

c) (2 points) Prove that in this vector space, scalar multiplication distributes over vector addition, i.e. that $r \otimes (x \oplus y) = (r \otimes x) \otimes (r \oplus y)$.

$$\begin{aligned} r \otimes (x \oplus y) &= r \otimes (xy) \\ &= (xy)^r \\ &= x^r y^r \\ &= (r \otimes x)(r \otimes y) = (r \otimes x) \oplus (r \otimes y) \end{aligned}$$