The determinant of a matrix

Convention: From now on all matrices are
square i.e. of size $n \times n$.

For any matrix $A \in M_{n\times n}$, we define a number $\det(A)$ or $|A|$ (the determinant of A).

It has this property:

$\det(A) \neq 0$ if and only if A^{-1} exists

To define it, we first need another definition:

Definition
The first minor A_{ij} of the matrix $A \in M_{n\times n}$ is the $(n-1)\times(n-1)$ matrix we get after deleting the i^{th} row and j^{th} column of A.

Example

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix} \qquad A_{1,1} = \begin{bmatrix} 0 & 5 \\ 9 & 11 \end{bmatrix}$$

$$A_{1,3} = \begin{bmatrix} 3 & 0 \\ -1 & 9 \end{bmatrix}$$

$A_{3,1} = \begin{bmatrix} 4 & 7 \\ 0 & 5 \end{bmatrix}$ $\quad A_{2,3} = \begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix}$

We define the determinant inductively, i.e. by first defining it for 1x1 matrices, then 2x2 matrices, then 3x3 matrices, etc.

1x1 matrices

If $A = [a_{1,1}]$ then $\det(A) = |A| = a_{1,1}$

2x2 matrices

If $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ then $\det(A) = |A| = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$

Cofactor expansion definition: We can also define the determinant by "expanding along a row" or "expanding along a column"

Say we pick <u>Row 1</u> then the cofactor expansion is

$$(-1)^{1+1} a_{1,1} |A_{1,1}| + (-1)^{1+2} a_{1,2} |A_{1,2}|$$

$$= a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

Say we pick column 2 then the cofactor expansion is

$$(-1)^{1+2} a_{1,2} |A_{1,2}| + (-1)^{2+2} a_{2,2} |A_{2,2}|$$

$$= -a_{1,2} a_{2,1} + a_{2,2} a_{1,1}$$

$$= a_{1,1} a_{2,2} - a_{1,2} a_{2,1}$$

It doesn't matter what row or column we pick, we always get the same answer.

3 x 3 matrices

There is a long explicit formula which no one knows. Starting here cofactor expansion is simpler.

Say $A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$

The cofactor expansion along row i is

$$(-1)^{i+1} a_{i,1} |A_{i,1}| + (-1)^{i+2} a_{i,2} |A_{i,2}| + (-1)^{i+3} a_{i,3} |A_{i,3}|$$

2x2 matrices whose determinants we know how to compute

Along <u>column</u> j the cofactor expansion is

$$(-1)^{1+j} a_{1,j} |A_{1,j}| + (-1)^{2+j} a_{2,j} |A_{2,j}| + (-1)^{3+j} a_{3,j} |A_{3,j}|$$

2x2 matrices

Again, it doesn't matter which row or column we use, they will all give the same answer

<u>n x n matrices</u>

It's the same idea

Cofactor expansion along row i:

$$\sum_{j=1}^{n} (-1)^{i+j} a_{i,j} |A_{i,j}|$$

$$= (-1)^{i+1} a_{i,1} |A_{i,1}| + (-1)^{i+2} a_{i,2} |A_{i,2}| + \dots + (-1)^{i+n} a_{i,n} |A_{i,n}|$$

Cofactor expansion along column j:

$$\sum_{i=1}^{n} (-1)^{i+j} a_{i,j} |A_{i,j}|$$

$$= (-1)^{1+j} a_{1,j} |A_{1,j}| + (-1)^{2+j} a_{2,j} |A_{2,j}| + \dots + (-1)^{n+j} a_{n,j} |A_{n,j}|$$

It never matters which row or column we use so we might as well pick one with many zeroes to make things easy.

Example: $A = \begin{bmatrix} 0 & 4 & 0 & -3 \\ 1 & 1 & 5 & 2 \\ 1 & -2 & 0 & 6 \\ 3 & 0 & 0 & 1 \end{bmatrix}$

this column has 3 zeroes!!

we expand along column 3

$$|A| = (-1)^{1+3} \cdot 0 \cdot |A_{1,3}| + (-1)^{2+3} \cdot 5 \cdot |A_{2,3}|$$
$$+ (-1)^{3+3} \cdot 0 \cdot |A_{3,3}| + (-1)^{4+3} \cdot 0 \cdot |A_{4,3}|$$

$$= -5 \begin{vmatrix} 0 & 4 & -3 \\ 1 & -2 & 6 \\ 3 & 0 & 1 \end{vmatrix}$$

← the best we can do is one zero, so let's expand along the first row
call this 3x3 matrix B

$$= -5 \left((-1)^{1+1} \cdot 0 \cdot |B_{1,1}| + (-1)^{1+2} \cdot 4 \cdot |B_{1,2}| + (-1)^{1+3} \cdot (-3) |B_{1,3}| \right)$$

$$= -5\left(-4\begin{vmatrix} 1 & 6 \\ 3 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix}\right)$$

$$= -5\left(-4(1\cdot 1 - 6\cdot 3) - 3(1\cdot 0 - (-2)\cdot 3)\right)$$

(for 2x2 matrices we use the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ because it's easy)

$$= -5(-4(1-18) - 3\cdot 6)$$

$$= -5(-4\cdot(-17) - 18)$$

$$= -5(68 - 18) = -5\cdot 50 = -250$$

Notes

- If A has a row or column of all zeroes, its determinant is 0 (expand along that row or column)
- Cofactor expansion is barbaric and takes forever. There is a much faster way to compute determinants by doing row operations (and keeping track of them!) until we have an easy matrix, then computing the easy determinant, then modifying to get the actual determinant. We won't have time to do this this semester

but it is explained in Section Four.I.2 and below.

- Actually in real life people use computers to compute determinants and there is no point getting really good at it by hand. The only reason we did cofactor expansion is that we will need it for the next topic.

Fast & easy way to compute determinants by hand (not covered in this class but I guess you could need this in another class?)

Steps

- get A in echelon form, write down every row operation you do! (this is important)
- in echelon form, the determinant is the product of the entries on the main diagonal
- modify the determinant of the matrix in echelon form in the following way:

-if you multiplied a row by k, divide the determinant by k (and if you divided a row by k, multiply the determinant by k)

- for every row swap, multiply the determinant by -1.

Example: $A = \begin{bmatrix} 0 & 4 & 0 & -3 \\ 1 & 1 & 5 & 2 \\ 1 & -2 & 0 & 6 \\ 3 & 0 & 0 & 1 \end{bmatrix}$

$$\overset{\rho_1 \leftrightarrow \rho_2}{\sim} \begin{bmatrix} 1 & 1 & 5 & 2 \\ 0 & 4 & 0 & -3 \\ 1 & -2 & 0 & 6 \\ 3 & 0 & 0 & 1 \end{bmatrix} \underset{\rho_4 - 3\rho_1}{\overset{\rho_3 - \rho_1}{\sim}} \begin{bmatrix} 1 & 1 & 5 & 2 \\ 0 & 4 & 0 & -3 \\ 0 & -3 & -5 & 4 \\ 0 & -3 & -15 & -5 \end{bmatrix}$$

$$\overset{\rho_4 - \rho_3}{\sim} \begin{bmatrix} 1 & 1 & 5 & 2 \\ 0 & 4 & 0 & -3 \\ 0 & -3 & -5 & 4 \\ 0 & 0 & -10 & -9 \end{bmatrix} \overset{\frac{1}{4}\rho_2}{\sim} \begin{bmatrix} 1 & 1 & 5 & 2 \\ 0 & 1 & 0 & -3/4 \\ 0 & -3 & -5 & 4 \\ 0 & 0 & -10 & -9 \end{bmatrix}$$

$$\overset{\rho_3 + 3\rho_2}{\sim} \begin{bmatrix} 1 & 1 & 5 & 2 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & -5 & 7/4 \\ 0 & 0 & -10 & -9 \end{bmatrix} \overset{\rho_4 - 2\rho_3}{\sim} \begin{bmatrix} 1 & 1 & 5 & 2 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & -5 & 7/4 \\ 0 & 0 & 0 & -25/2 \end{bmatrix}$$

Now $\begin{vmatrix} 1 & 1 & 5 & 2 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & -5 & -7/4 \\ 0 & 0 & 0 & -25/2 \end{vmatrix} = 1\cdot 1\cdot(-5)\cdot\left(\frac{-25}{2}\right) = \frac{125}{2}$

(expand along column 1, then along 1st column of minor; this is how this always works)

And we did:

- $\rho_1 \leftrightarrow \rho_2$ so $\frac{125}{2}$ becomes $-\frac{125}{2}$
- $\frac{1}{4}\rho_2$ so $-\frac{125}{2}$ becomes $4\cdot\frac{-125}{2} = -250$

(we ignore the other row operations)

OK, this wasn't so much better but usually it is.

<u>Warning</u> If you know the "diagonals" trick for 3x3 matrices, do <u>not</u> try to apply it to larger matrices, it does <u>not</u> work!!