Change of basis

Most of the time, the standard basis for $\mathbb{R}^n$ is great. However, when n is really large computations can get slow and we might want to use a different basis to speed things up.

Example $f:\mathbb{R}^3 \to \mathbb{R}^3$ given by $\begin{pmatrix} 1 & 5 & 0 \\ 1 & 5 & 0 \\ 1 & -1 & 6 \end{pmatrix}$

To compute $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+5y \\ x+5y \\ x-y+6z \end{pmatrix}$ require 3 multiplications and 4 additions/subtractions

(This isn't so bad but if n was big it would get worse)

This is because we insist on writing

$$\vec{v} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

standard basis

Consider instead the basis

$$B = \left\{ \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(I got this from the eigenvalues notes, page 9; you can check it's a basis)

What is neat about this basis is that they are all eigenvectors of f:

$$f\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} \qquad \left(= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ of course}\right)$$

$$f\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 6\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

So if $\vec{v} = a_1\begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} + a_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(a_1, a_2, a_3 exist and are unique because B is a basis) then

$$f(\vec{v}) = f\left(a_1\begin{pmatrix}-5\\1\\1\end{pmatrix} + a_2\begin{pmatrix}1\\1\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= a_1 f\begin{pmatrix}-5\\1\\1\end{pmatrix} + a_2 f\begin{pmatrix}1\\1\\0\end{pmatrix} + a_3 f\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= a_1 \cdot 0\begin{pmatrix}-5\\1\\1\end{pmatrix} + a_2 \cdot 6\begin{pmatrix}1\\1\\0\end{pmatrix} + a_3 \cdot 6\begin{pmatrix}0\\0\\1\end{pmatrix}$$

i.e. if $\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$

then $\mathrm{Rep}_B(f(\vec{v})) = \begin{pmatrix}0\\6a_2\\6a_3\end{pmatrix}$ This is 2 multiplications only!

It might then be advantageous to use the basis B instead of the standard basis if we have to compute $f(\vec{v})$ often.

Recall that $f: \mathbb{R}^n \to \mathbb{R}^n$ is

<u>diagonalizable</u> if by putting together a basis for each eigenspace we get a basis for all of $\mathbb{R}^n$.

Say that f has eigenvalues

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$$

with eigenvectors

$$\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n$$

where
- for each i $\quad f(\vec{v}_i) = \lambda_i \vec{v}_i$
- the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent
- the λ_i's might not be all different

Example: for $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+5y \\ x+5y \\ x-y+6z \end{pmatrix}$ we could write

$$\lambda_1 = 0, \ \vec{v}_1 = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}; \ \lambda_2 = 6, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

$\lambda_3 = 6, \ \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Then if $B = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$

$$\mathrm{Rep}_B(f(\vec{v})) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \mathrm{Rep}_B(\vec{v})$$

matrix with eigenvalues on diagonal and zeroes elsewhere

This explains why we say f is diagonalizable:
There is a basis for which the action of f is given by a diagonal matrix.

Change of basis matrix

To use this, all that remains to be done is to quickly go between $\vec{v}$ in the standard basis and $\vec{v}$ in the eigenvector basis B.

We will use our example to show how this works:

$$B = \left\{ \vec{v}_1 = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{if}$$

$$\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3$$

Say $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ To find $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ we solve

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = a_1 \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

OR

$$\begin{aligned} 1 &= -5a_1 + a_2 \\ 2 &= a_1 + a_2 \\ 3 &= a_1 \qquad + a_3 \end{aligned}$$

OR

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

If $C = \begin{pmatrix} -5 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (C is for change of basis)

then $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = C^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Compute C^{-1}:

$$\left(\begin{array}{ccc|ccc} -5 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{\rho_2 \leftrightarrow \rho_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ -5 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \rho_2 + 5\rho_1 \\ \sim \\ \rho_3 - \rho_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 1 & 5 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{\rho_2 \leftrightarrow -\rho_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 6 & 0 & 1 & 5 & 0 \end{array}\right)$$

$$\begin{array}{c} \rho_1 - \rho_2 \\ \sim \\ \rho_3 - 6\rho_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 6 & 1 & -1 & 6 \end{array}\right)$$

$$\overset{\frac{1}{6}\rho_3}{\sim}\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{6} & 1 \end{array}\right)$$

$$\underset{\rho_2+\rho_3}{\overset{\rho_1-\rho_3}{\sim}}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{5}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{6} & 1 \end{array}\right)$$

so $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & -\frac{5}{6} & 0 \\ \frac{1}{6} & -\frac{1}{6} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 1/6 \\ -3/2 \\ 17/6 \end{pmatrix}$$

So if $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then

$$\mathrm{Rep}_B(f(\vec{v})) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 1/6 \\ -3/2 \\ 17/6 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \\ 17 \end{pmatrix}$$

What if we want $f(\vec{v})$ in the standard basis?

Just do

$$f(\vec{v}) = 0\begin{pmatrix}-5\\1\\1\end{pmatrix} + (-9)\begin{pmatrix}1\\1\\0\end{pmatrix} + 17\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}-5 & 1 & 0\\1 & 1 & 0\\1 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\-9\\17\end{pmatrix}$$

↑ this is the matrix C!

So C makes the change of basis $B \longrightarrow$ standard

C^{-1} makes the change of basis standard $\rightarrow B$

This looks like a lot of work but it can help.

Example:

Let $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2x+6y \\ 3x+2y+z \end{pmatrix}$

Let $\vec{v} = \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix}$. Compute $f(f(f(f(f(\vec{v})))))$.

This is some work but it's not too bad to do once. If you had to do it several times though, here is what you would do:

Step 1: Find a basis of eigenvectors

The matrix associated to f is $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

Its characteristic polynomial is

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 2 & 6-\lambda & 0 \\ 3 & 2 & 1-\lambda \end{vmatrix}$$ ← expand along this row

$$= (-1)^{1+1}(2-\lambda)\begin{vmatrix} 6-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(6-\lambda)(1-\lambda)$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 6$, $\lambda_3 = 1$

$\underline{\lambda_1 = 2}$ Solve $(A-2I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 2 & -1 \end{pmatrix} \overset{\rho_1 \leftrightarrow \frac{1}{2}\rho_2}{\sim} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 3 & 2 & -1 \end{pmatrix} \overset{\rho_3 - 3\rho_1}{\sim} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & -1 \end{pmatrix}$$

$$\overset{-\frac{1}{4}\rho_3 \leftrightarrow \rho_2}{\sim} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{pmatrix} \overset{\rho_1 - 2\rho_2}{\sim} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x - \tfrac{1}{2}z &= 0 & x &= \tfrac{1}{2}z \\ y + \tfrac{1}{4}z &= 0 & y &= -\tfrac{1}{4}z \end{aligned} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/4 \\ 1 \end{pmatrix} z$$

$$\vec{v}_1 = \begin{pmatrix} 1/2 \\ -1/4 \\ 1 \end{pmatrix}$$

$\underline{\lambda_2 = 6}$ Solve $(A-6I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & -5 \end{pmatrix} \underset{\frac{1}{2}\rho_2}{\overset{-\frac{1}{4}\rho_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & -5 \end{pmatrix} \underset{\rho_3 - 3\rho_1}{\overset{\rho_2 - \rho_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -5 \end{pmatrix}$$

$x = 0$

$2y - 5z = 0 \quad y = \frac{5}{2}z \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5/2 \\ 1 \end{pmatrix} z$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 5/2 \\ 1 \end{pmatrix}$$

$\underline{\underline{\lambda_3 = 1}}$ Solve $(A - I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 2 & 0 \end{pmatrix} \underset{\rho_3 - 3\rho_1}{\overset{\rho_2 - 2\rho_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{pmatrix} \overset{\rho_3 - \frac{2}{5}\rho_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x = 0$

$5y = 0 \to y = 0 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z$

z is free

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We join the bases of the 3 eigenspaces to get a basis for $\mathbb{R}^3$:

$$B = \left\{ \begin{pmatrix} 1/2 \\ -1/4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 5/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Step ② Compute the change of basis matrices

We know $$C = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/4 & 5/2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

goes from B to standard basis

Compute C^{-1} to go from standard to B:

$$\left(\begin{array}{ccc|ccc} 1/2 & 0 & 0 & 1 & 0 & 0 \\ -1/4 & 5/2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} 2\rho_1 \\ \sim \\ 4\rho_2 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ -1 & 10 & 0 & 0 & 4 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{matrix} \rho_2+\rho_1 \\ \sim \\ \rho_3-\rho_1 \end{matrix} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 10 & 0 & 2 & 4 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right) \overset{\frac{1}{10}\rho_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1/5 & 2/5 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\overset{\rho_3-\rho_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1/5 & 2/5 & 0 \\ 0 & 0 & 1 & -11/5 & -2/5 & 1 \end{array}\right)$$

$$C^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 1/5 & 2/5 & 0 \\ -11/5 & -2/5 & 1 \end{pmatrix}$$

Step 3 Finally we compute!

Instead of doing f 5 times in the standard basis, we

a) write $\vec{v}$ in the basis B

$$\mathrm{Rep}_B(\vec{v}) = C^{-1}\vec{v} = \begin{pmatrix} 2 & 0 & 0 \\ 1/5 & 2/5 & 0 \\ -11/5 & -2/5 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

b) Do f 5 times <u>in the B basis</u>

In the B basis f is just $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} = C^{-1}AC$

$$\begin{tikzcd} \underset{\text{standard}}{\mathbb{R}^3} \arrow[r, "A"] \arrow[d, "C^{-1}"'] & \underset{\text{standard}}{\mathbb{R}^3} \\ \underset{\text{B basis}}{\mathbb{R}^3} \arrow[r, "\left(\begin{smallmatrix} 2&0&0\\0&6&0\\0&0&1 \end{smallmatrix}\right)"] & \underset{\text{B basis}}{\mathbb{R}^3} \arrow[u, "C"'] \end{tikzcd}$$

f once: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 3 \end{pmatrix}$

twice: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 8 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 16 \\ 0 \\ 3 \end{pmatrix}$

three times $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 16 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 0 \\ 3 \end{pmatrix}$

four times $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 32 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 64 \\ 0 \\ 3 \end{pmatrix}$

five times $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 64 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 128 \\ 0 \\ 3 \end{pmatrix}$

(or just do $\begin{pmatrix} 2^5 \cdot 4 \\ 6^5 \cdot 0 \\ 1^5 \cdot 3 \end{pmatrix} = \begin{pmatrix} 128 \\ 0 \\ 3 \end{pmatrix}$

c) Go back to standard basis

$$\begin{pmatrix} 1/2 & 0 & 0 \\ -1/4 & 5/2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 128 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 64 \\ -32 \\ 131 \end{pmatrix}$$

so $f(f(f(f(f\begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix})))) = \begin{pmatrix} 64 \\ -32 \\ 131 \end{pmatrix}$

Something you might need to do:

Go from basis B to basis D (neither of which are standard).

Easy! Do

$$B \xrightarrow{C_B} \text{standard} \xrightarrow{C_D^{-1}} D$$

i.e. the matrix multiplication

$$C_D^{-1} C_B \vec{v}$$

(remember that $C_D^{-1} C_B$ means "first do C_B then do C_D^{-1}")

What if f is not diagonalizable?

We can still do something pretty good using the Jordan canonical form.