Problem 1 (10 points): Find the general solution of the system

\[
\mathbf{x}'(t) = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix} \mathbf{x}(t)
\]

Solution: We first find the characteristic polynomial of this matrix:

\[
\begin{vmatrix} -2 - \lambda & -9 & 0 \\ 1 & 4 - \lambda & 0 \\ 1 & 3 & 1 - \lambda \end{vmatrix} = (1 - \lambda)[(-2 - \lambda)(4 - \lambda) + 9] = (1 - \lambda)(\lambda - 1)^2
\]

So we have the eigenvalue \( \lambda = 1 \) repeated three times. We now find the dimension of the eigenspace:

\[
\begin{bmatrix} -3 & 9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

From this we see that the eigenspace is two dimensional, so we are looking for two linearly independent eigenvectors \( \mathbf{v}_1 \) and \( \mathbf{u}_1 \), and a generalized eigenvector \( \mathbf{v}_2 \) such that \((\mathbf{A} - \mathbf{I})^2\mathbf{v}_2 = 0\) and \((\mathbf{A} - \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1\).

First we find that \((\mathbf{A} - \mathbf{I})^2 = 0\), so we try \( \mathbf{v}_2 = (1,0,0) \). This gives us \((\mathbf{A} - \mathbf{I})\mathbf{v}_2 = (-3,1,1) = \mathbf{v}_1\).

We now need to find another linearly independent eigenvector \( \mathbf{u}_1 \). We find the eigenspace to be

\[
\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s
\]

Our vector \( \mathbf{v}_1 \) is

\[
\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

(in other words \( t = 1 \) and \( s = 1 \)), so one of many choices for \( \mathbf{u}_1 \) is \((0,0,1) \) (\( t = 0 \) and \( s = 1 \)).

So the general solution is

\[
\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + c_3 e^t \left( \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)
\]