NAME:

To get the most of this quiz, allow yourself no more than 20 minutes to completely answer it, and do not use any notes or outside help. I will grade it if you hand it in during discussion on April 27 or 29.

Problem 1 (10 points): *Find the general solution to the differential equation*

$$x^2y'' + 3xy' + y = x^{-1}, \quad x > 0$$

Solution: The homogeneous equation that corresponds to this differential equation is an Euler equation, and so we must use variation of parameters to solve this. We first solve the homogeneous problem.

The equation

$$x^2y'' + 3xy' + y = 0, \quad x > 0$$

has characteristic equation

$$r(r-1) + 3r + 1 = r^2 + 2r + 1 = (r+1)^2 = 0$$

so the solution is $y(x) = c_1x^{-1} + c_2x^{-1}\ln x$. (The natural logarithm doesn't have absolute values because of the condition $x > 0$.)

We now set up our variation of parameters. Here we have $y_1 = x^{-1}$ and $y_2 = x^{-1}\ln x$, so that $y_1' = -x^{-2}$ and $y_2' = -x^{-2}\ln x + x^{-2}$. So we set up the linear system we need to solve:

$$\begin{bmatrix} x^{-1} & x^{-1}\ln x \\ -x^{-2} & -x^{-2}\ln x + x^{-2} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x^{-3} \end{bmatrix}$$

And now we solve:

$$\begin{bmatrix} x^{-1} & x^{-1}\ln x & 0 \\ -x^{-2} & -x^{-2}\ln x + x^{-2} & x^{-3} \end{bmatrix} \sim \begin{bmatrix} 1 & \ln x & 0 \\ -1 & -\ln x + 1 & x^{-1} \end{bmatrix} \sim \begin{bmatrix} 1 & \ln x & 0 \\ 0 & 1 & x^{-1} \end{bmatrix}$$

so that $u_2' = x^{-1}$ and $u_1' = -\frac{\ln x}{x}$. We now solve for u_1 and u_2:

$$u_1 = -\int \frac{\ln x}{x}\,dx = -\int u\,du = -\frac{u^2}{2} + C = -\frac{(\ln x)^2}{2} + C$$

(This was done using u-substitution.)

$$u_2 = \int x^{-1}\,dx = \ln x + D$$

So the general solution is

$$\begin{aligned} y(x) &= \left(-\frac{(\ln x)^2}{2} + C\right)x^{-1} + (\ln x + D)\,x^{-1}\ln x \\ &= c_1x^{-1} + c_2x^{-1}\ln x + \frac{x^{-1}(\ln x)^2}{2}. \end{aligned}$$